A GRADUATED SERIES OF EXERCISES IN ELEMENTARY ALGEBRA, WITH APPENDICES, CONTAINING PAPERS OF MISCELLANEOUS EXAMPLES.

Designed for the Use of Schools.

BY

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ADVERTISEMENT.

To this issue I have prefixed questions on the theory of Vulgar and Decimal Fractions, and have added a second series of Miscellaneous Examples of a somewhat more difficult character. These, and a few other additions, though made for a special purpose, may, it is hoped, render the book more generally acceptable to those who find it suited to their needs, either as a companion to Mr. Lund's Easy Algebra, or as a syllabus and exercise book for a system of oral teaching.

G. F. W.

CONTENTS.

ARITHMETICAL FRACTIONS.

1. EXPLAIN fully what is meant by the following: $\frac{3}{5}$ of an orange, $\frac{15}{8}$ of an hour, $4\frac{3}{7}$ yards.

2. By what names are the three kinds of fractional magnitudes $\frac{3}{5}$, $\frac{15}{8}$, $4\frac{3}{7}$ distinguished?

3. Show by reference to a divided line (such as a carpenter's rule) that *three-fourths* of *one* is the same as *one-fourth* of *three.*

4. Show in a similar way that $\frac{2}{3} = \frac{4}{6} = \frac{8}{12} =$ &c.;

and also $= \frac{6}{9} = \frac{18}{27} =$ &c.

What general propositions may be inferred from these results?

5. Write down a *proper* fraction, an *improper* fraction, and a *mixed quantity.*

6. Show fully that $2\frac{5}{7} = \frac{19}{7}$. Thence deduce the rule for turning mixed quantities into improper fractions, and conversely.

7. Write down a number of fractions each equal to $\frac{3}{4}$, $\frac{5}{7}$.

8. Write down all the simpler fractions equivalent to $\frac{48}{120}$.

9. Reduce the following to lowest terms: $\frac{8}{32}$, $\frac{12}{56}$, $\frac{18}{27}$, $\frac{14}{42}$, $\frac{54}{464}$, $\frac{72}{574}$.

10. Change the fractions $\frac{7}{12}$, $\frac{3}{8}$, $\frac{7}{15}$, $\frac{17}{40}$, $\frac{4}{15}$, $\frac{5}{28}$, into equivalents with denominators 72, 360, 210, 240, 120, 336.

11. Before fractions can be added together they must generally be modified in form. Why and how? Explain by examples.

In practice, the *least* common denominator is always sought: how is this found?

12. How may the values of two or more proposed fractions be compared?

13. State fully the rule for subtracting one fraction from another.

14. Explain the following:

$$(1.)\quad \tfrac{3}{7} \times 5 = 2\tfrac{1}{7}.$$

$$(2.)\quad \left.\begin{array}{l} \tfrac{7}{12} \times 3 = \tfrac{7}{4} \\ \tfrac{5}{24} \times 9 = \tfrac{15}{8} \end{array}\right\}.$$

Infer from (1) the rule for multiplying a fraction by a whole number. What modifications of the rule do the results (2) point out?

15. Deduce from (1), (2) the rules for dividing a fraction by a whole number. Establish your rules also without reference to those for multiplication.

16. Show that $\frac{3}{5}$ of $\frac{4}{7} = \frac{12}{35}$.

17. State the rules for the so-called multiplication of fractions. Hence show that $\frac{3}{5} \times \frac{4}{7}$ means $\frac{3}{5}$ of $\frac{4}{7}$; and $\frac{12}{35} \div \frac{4}{7}$ means $\frac{7}{4}$ of $\frac{12}{35}$.

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18. Reduce to lowest terms: $\frac{7}{21}$, $\frac{13}{65}$, $\frac{20}{28}$, $\frac{68}{1428}$, $\frac{391}{437}$, $\frac{832}{1088}$, $\frac{6127}{6721}$, $\frac{6902}{8265}$, $\frac{7876}{9681}$.

19. Reduce to improper fractions: $7\frac{3}{5}$, $11\frac{7}{13}$, $8\frac{9}{14}$, $21\frac{7}{15}$, $3\frac{40}{51}$, $19\frac{17}{18}$, $65\frac{3}{61}$.

20. Reduce to simplest mixed numbers: $\frac{20}{17}$, $\frac{52}{12}$, $\frac{99}{18}$, $\frac{45}{23}$, $\frac{167}{142}$, $\frac{3235}{122}$, $\frac{62366}{232}$.

21. Express by simple fractions:

$\frac{1}{3}$ of $\frac{1}{2}$; $\frac{3}{5}$ of $\frac{1}{4}$; $\frac{7}{8}$ of $\frac{3}{7}$; $4\frac{2}{3}$ of $3\frac{3}{5}$; $1\frac{3}{5}$ of $2\frac{5}{12}$ of $\frac{10}{29}$;
$\frac{1}{2}$ of $\frac{3}{4}$ of $9\frac{4}{5}$ of $\frac{16}{39}$ of $\frac{2}{3}$.

22. Compare the following sets of fractions:

$\frac{5}{8}$, $\frac{7}{12}$, $\frac{8}{9}$; $\frac{3}{5}$, $\frac{2}{15}$, $\frac{33}{40}$, $\frac{12}{21}$; $\frac{3}{14}$, $\frac{13}{15}$, $\frac{17}{70}$, $\frac{61}{65}$, $\frac{37}{45}$.

23. Find the values in simplest forms of the following:

(1.) $\frac{1}{2} + \frac{2}{3} + \frac{3}{5}$; $\frac{7}{9} + \frac{10}{13} + \frac{9}{16}$; $\frac{14}{17} + \frac{19}{21}$.

(2.) $\frac{1}{3} + \frac{7}{9}$; $\frac{11}{16} + \frac{5}{64} + \frac{13}{128}$; $\frac{73}{164} + \frac{32}{41}$.

(3.) $3\frac{2}{5} + \frac{5}{8} + \frac{9}{14} + 4\frac{33}{35} + \frac{4}{15} + \frac{16}{21}$; $17\frac{8}{15} + 8\frac{14}{25}$.

(4.) $\frac{7}{13} - \frac{23}{65}$; $5\frac{41}{45} - 3\frac{21}{25}$; $19 - 5\frac{81}{83}$; $17\frac{13}{27} - 8\frac{12}{25}$.

(5.) $\frac{4}{5} + \frac{13}{20} - \frac{17}{35}$; $\frac{25}{32} + \frac{11}{16} - \frac{17}{18} + \frac{9}{14}$; $\frac{54}{91} + \frac{67}{68} + \frac{39}{42} - \frac{25}{36}$.

(6.) $1-\frac{1}{2}+\frac{2}{3}-\frac{3}{4}+\frac{4}{5}-\frac{5}{6}+\frac{6}{7}-\frac{7}{8}+\frac{8}{9}-\frac{9}{10}+\frac{10}{11}-\frac{11}{12}$.

(7.) $\frac{3}{5}\times 7$; $\frac{5}{12}\times 6$; $\frac{10}{13}\times 11$; $\frac{18}{25}\times 15$; $\frac{28}{45}\times 42$; $3\frac{4}{5}\times 3$.

(8.) $\frac{9}{13}\div 3$; $\frac{7}{16}\div 7$; $\frac{4}{23}\div 12$; $\frac{27}{31}\div 18$; $\frac{18}{53}\div 23$; $\frac{35}{72}\div 36$.

(9.) $\frac{1}{3}\times\frac{1}{2}$; $\frac{2}{5}\times\frac{1}{4}$; $\frac{3}{4}\times\frac{1}{5}$; $\frac{5}{12}\times\frac{3}{5}$; $1\frac{7}{13}\times\frac{7}{8}$; $3\frac{9}{11}\times 2\frac{4}{13}$.

(10.) $\frac{1}{6}\div\frac{1}{2}$; $\frac{3}{10}\div\frac{2}{5}$; $\frac{2}{9}\div\frac{3}{7}$; $6\frac{15}{16}\div 4\frac{5}{8}$; $25\frac{7}{8}\div 34\frac{11}{16}$.

24. Simplify the following:

(1.) $2\frac{1}{3}\times 3\frac{1}{4}\times\frac{1}{7}\times 5\frac{1}{13}$; $\frac{3}{8}$ of $\frac{9}{10}+\frac{17}{20}$; $\frac{3}{8}$ of $(\frac{9}{10}+\frac{17}{20})$.

(2.) $3\frac{5}{6}$ of $2\frac{4}{23}$ of $\frac{7}{25}$; $\frac{2}{5}$ of $\frac{3}{7}-\frac{4}{70}$; $\frac{2}{5}$ of $(\frac{3}{7}-\frac{4}{70})$.

(3.) $\dfrac{9\frac{7}{9}}{12\frac{5}{6}}$; $\dfrac{1\frac{1}{3}}{2\frac{2}{3}}$ of $\dfrac{3\frac{1}{2}}{21}$; $(\frac{5}{14}$ of $\frac{2}{9}$ of $13)\div\frac{2}{9}$ of $\dfrac{7\frac{1}{2}}{8\frac{1}{3}}$.

(4.) $\dfrac{3\frac{1}{9}}{\frac{3}{4}+1\frac{7}{12}}$; $\dfrac{2\frac{1}{2}\times 2\frac{1}{2}-1}{2\frac{1}{2}\times 2\frac{1}{2}+1}$; $\dfrac{\frac{19}{20}+3}{\frac{19}{20}\text{ of }3}\div\frac{79}{57}$.

(5.) $\dfrac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{1}{2\frac{1}{2}}+\frac{1}{3\frac{1}{2}}+\frac{1}{4\frac{1}{2}}}$; $\dfrac{1}{2+\dfrac{1}{3+\frac{1}{4}}}$; $\dfrac{\frac{2}{3}\times\frac{3}{4}+\frac{1}{2}\div 2\frac{1}{2}}{(\frac{3}{4}-\frac{1}{3})\div(\frac{1}{2}+\frac{2}{3}\text{ of }6)}$

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1. What are decimal fractions? Explain the mode in which they are written, and the effect of changing the position of the decimal point.

2. Show that $\frac{3}{5}=\cdot 6$; $\frac{5}{16}=\cdot 3125$; $\frac{6}{7}=\cdot 85714\ldots$

When can a vulgar fraction be expressed exactly as a decimal? When only approximately?

3. Convert $\frac{1}{9}$, $\frac{1}{99}$, $\frac{1}{999}$, $\frac{1}{9999}$, &c. into decimals; and hence deduce rules for converting circulating decimals, both pure and mixed, into their equivalent vulgar fractions.

4. Add together 63·507 and 7·412, and subtract the latter from the former, giving reasons for your method.

5. State, with reasons, the rule for the multiplication of decimals.

6. Divide ·336 by 42; 3·36 by 4·2; and 33·6 by ·42; giving full explanations regarding the position of the decimal point.

State a rule for the division of decimals.

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7. What multiple or what part is each of the following decimals of the one which follows it:

10·23, 1·023, 102·3, ·01023, 1023, ·1023, ·0001023?

8. Convert into decimals: $\frac{1}{8}$, $\frac{2}{25}$, $\frac{3}{7}$, $\frac{5}{14}$, $\frac{9}{16}$, $\frac{31}{75}$, $\frac{16}{81}$, $\frac{14}{99}$, $\frac{42}{167}$.

9. Convert into vulgar fractions: ·6, ·12, ·75, ·374, ·0135, ·776, ·666, $\cdot 2121\dot{2}\dot{1}$, $\cdot 423\dot{2}\dot{3}$, $\cdot 145\dot{1}4\dot{5}$, $\cdot 02\dot{1}4\dot{5}$.

10. Find the value of the following:

(1.) $27{\cdot}61 + 398{\cdot}5 + 2{\cdot}304 + {\cdot}00125$; $18{\cdot}321 - 13{\cdot}0025$.

(2.) $17{\cdot}62 \times 9$; $1{\cdot}762 \times {\cdot}09$; ${\cdot}012 \times {\cdot}03$; $120 \times 45{\cdot}03$; $43{\cdot}2 \times 19$.

(3.) $62{\cdot}5 \div 25$; $6{\cdot}25 \div 2{\cdot}5$; ${\cdot}625 \div {\cdot}25$; $6{\cdot}25 \div {\cdot}0025$; $6{\cdot}25 \div {\cdot}25$.

(4.) $62{\cdot}5 \div 25000$; $25{\cdot}132 \div 6{\cdot}2835$; $196{\cdot}5 \div 61{\cdot}417$; $44{\cdot}2854 \div {\cdot}278$.

(5.) ${\cdot}034 \div 2{\cdot}14$; $12{\cdot}34 \div {\cdot}000027$; ${\cdot}00035 \div 250$; ${\cdot}000785 \div {\cdot}0005$.

11. Convert into decimals, accurately to five places:

$$2 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{2 \times 3 \times 4 \times 5} + \text{\&c.} \ldots .$$

A

GRADUATED SERIES OF EXERCISES

IN

ELEMENTARY ALGEBRA.

NOTATION. FIRST PRINCIPLES.

EXERCISE I.

1. THE instruments of Algebraical Calculation are *letters* and *signs*. What are the *letters* used to express? What are the principal *signs* now in use?

2. Explain the meaning of the signs $+$, $-$, &c., as they occur in the following algebraical expressions; $a + b$, $c - d$, $a \backsim b$, $m \times n$, $m \, . \, n$, $p \div q$. What do mn and $\frac{p}{q}$ signify? Write these expressions in *words* as they should be read.

3. Define the terms *factor*, *coefficient*. What are the factors of 15? What are the coefficients of 5 and 3 respectively in it? What are the coefficients of a in $3a$, $2a$, a?

4. The product of two algebraical quantities a and b may be expressed in three ways: (1.) $a \times b$, (2.) $a \, . \, b$, (3.) ab. If in place of a and b we write specific numbers (say 4 and 9), which of these ways should be used, and why should the other two be avoided?

5. A whole number and a fraction, such as $4\frac{3}{7}$, is in Arithmetic called a mixed number. If a stand for the whole number, b for the numerator, and c for the denominator of the fraction, would a mixed number be represented generally in Algebra by $a\,\frac{b}{c}$; and, if not, how should it be written, and what does $a\,\frac{b}{c}$ mean?

6. Explain the meaning of the signs in the expressions $a = b$, $c > d$, $e < f$. $\because\ 3x = 18$ $\therefore\ x = 6$.

Write these expressions in *words* as they should be read.

7. Express by means of algebraical symbols the *sum* of any two quantities, their *product* and *quotient*. Indicate that one quantity is to be subtracted from another: also the proposition, "The result produced by adding any quantity to its double is equal to three times that quantity."

8. Define the terms *product*, *quotient*. Indicate that one quantity is to be multiplied by two others; that one quantity is to be divided by the product of two others: also the proposition that "the quotient of one quantity by a second, multiplied by their product, equals the product of the first multiplied by itself."

Exercise II.

1. If $a = 21$, $b = 8$, and $c = 5$, find the value of each of the following quantities:

(1.) $a + b$.	(8.) $a - c + b$.	(15.) $2a + 3b$.
(2.) $a - b$.	(9.) $a + c - b$.	(16.) $4a - 3b$.
(3.) $b + a$.	(10.) $b + a + c$.	(17.) $5b + 6a$.
(4.) $a + b + c$.	(11.) $b + a - c$.	(18.) $a + 2b + c + 9$.
(5.) $a + b - c$.	(12.) $c + b + a$.	(19.) $2a + b + 3c - 17$.
(6.) $a - b + c$.	(13.) $c - b + a$.	(20.) $5a + 3b - 4c$.
(7.) $a - b - c$.	(14.) $c + a - b$.	

2. Write down the meaning of $a + b + c$, and $a + b - c$, at full length in words.

3. In Question (1), which of the proposed quantities are of the same value as (5)? which of the same value as (6)? Explain how it is that some of these quantities, though differing in form, are of the same value, whilst others differing in form, as (5) and (6), are different in value.

4. What is the coefficient of a in (15), of b in (17), of c in (19)?

5. In the first fourteen of the above expressions a appears without a coefficient: does this mean that its coefficient is 0? if not, what is the coefficient of a, and why need it not be written?

What would be the value of a quantity whose coefficient is 0; for instance, ma, when m is supposed $= 0$?

Exercise III.

1. If $a = 9$, and b stand successively for 6, 7, 8, 9, the values of $a - b$ are 3, 2, 1, 0: how is its value represented when $b = 10$? What is the quantity $a - b$ called for this and all values of b greater than 9?

2. Give some illustration of the meaning of a negative quantity; by reference, for example, to a man's property and debts.

If a stand for 8, b for 3, c for 5, find the value of the following quantities:

(1.) $2a - 6b$. (3.) $3a - 10b + 6$. (5.) $9 - 12b + 4a$.

(2.) $a + 2b - 3c$. (4.) $4a - 12b + 9$. (6.) $2a - 50 - \frac{1}{5}c$.

3. What is meant by "*the sign*" of an algebraical quantity? What is the sign of a in $a - b$, and why is it not written?

4. What is meant by a *term?* Write in as many ways as possible, by changing the order of the terms, the quantities

$a + b$, $a - b$, $a + b - c$, $a - b - c$. Does a difference in order make a difference in value?

5. What are the quantities a, b, c, called in reference to the quantity abc? Does the order of the letters make any difference in the value of this product? Write it in as many ways as possible. How does $\frac{a}{b}$ differ from $\frac{b}{a}$?

6. What are the coefficients of xy, y, and x, in the quantity $3xy$? Is it wrong to write this quantity $x3y$, or $xy3$; and if not, why is it never so written?

7. If $a = 10$, $b = 2$, $x = 9$, show that abx is equal to axb and bxa. Also find the value of each of the following quantities:

(1.) $3ax$, $4bx$.
(2.) $10abx + xx$.
(3.) $ab + ax + bx$.
(4.) $2ab + 3ax - bx$.
(5.) $9ax + 5bx - 51bb$.
(6.) $abx \div 3$, $7abx \div 2x$.
(7.) $\frac{3aabx}{5abx}$, $\frac{4x + a}{b}$.
(8.) $\frac{2a - 2x}{b}$, $\frac{2a}{b} - \frac{2x}{b}$.
(9.) $\frac{2ax + 3b + 14}{2a}$, $\frac{5ab}{2b} + \frac{a}{b} - \frac{45}{x}$.
(10.) $\frac{a}{b} - 2 + \frac{2bx - 4}{4b}$.
(11.) $\frac{5a + x - 2}{b + 1} - \frac{8b + 9}{a - 5}$.
(12.) $\frac{3x}{4b + a - x} + \frac{a - 2b}{x - 3}$.

Which is greater, $\frac{a}{b} - 2$, or $\frac{a - 2}{b}$?

8. What is the coefficient of x in $\frac{3}{4}x$, of y in $\frac{3}{5}xy$, of z in $3axz$; also of a and x in $3axz$?

$\frac{3}{4}x$ is equal to $\frac{3x}{4}$: verify this by taking $x = 12$.

9. What are the factors of $3axz$? Write down the quantities whose factors are x, 2, y, 7, z; and 3, a, x, 2, b.

10. Write down in algebraical signs *three times* a *plus* z; 4x *times* y *minus* b; 5 *times* xy *plus* z; 5 *times* xy *multiplied by* z.

EXERCISE IV.

1. Write at full length the following abbreviations:

$$a^2,\ a^5,\ a^3b,\ a^2b^2,\ ab^3,\ ab^2c^3,\ a^3b^2c.$$

Write in words as they should be read, the quantities a^2, a^3, a^5, a^m. What does a^m mean?

2. Write in the usual abbreviated way, aaa, abb, $aabbb$, $aabccc$, $aaaaabbc$, $xyyzzzz$.

Distinguish between a^2 and $2a$.

3. What does $\sqrt{a}$ mean? What is it equal to when $a = 16$?

What does $\sqrt[3]{a}$ mean? and what is its value when $a = 27$?

Show, by taking a to stand for 64, that $\sqrt[3]{a}$ and $3\sqrt{a}$ do not mean the same thing.

4. What does $\sqrt{ab}$ mean? What $\sqrt{a}\,.\,b$? Show that when $a = 9$, and $b = 4$, one of these quantities is twice as great as the other. Write these quantities in words as they should be read.

5. What does $\sqrt{\frac{a}{b}}$ signify? What $\frac{\sqrt{a}}{b}$? Find their values when a is 36 and b is 4.

6. What is meant by $\sqrt{a+b}$, and how does it differ from $\sqrt{a} + b$? What are their values when $a = 9$, $b = 16$?

7. If a stand for 3, b for 4, c for 8, find the value of each of the following quantities:

(1.) a^2, a^3b, ab^2, b^3, ab^2c.

(2.) $\frac{a^2b^2}{c}$, $\frac{b^2}{c}$, $\frac{c^2}{b}$, $\frac{3bc^2}{4ac}$.

(3.) $\sqrt{b}$, $\sqrt[3]{c}$, $\sqrt{a+1}$, $\sqrt{7b-a}$, $\sqrt{a^2+b^2}$, $\sqrt{ac+3b}$, $\sqrt[3]{9a}$.

EXERCISE V.

OBSERVATION.—When an operation is indicated as to be performed on compound quantities enclosed within brackets, or placed

under a line, (called a *Vinculum*,) every such quantity, whether of two or more terms or factors, must be regarded as a *whole*.

The student must keep this constantly in mind, as it is a very common error with beginners to suppose that the brackets merely imply that the operation indicated is to be performed on every term within the bracket alike. Thus it is often thought that, because a^2 signifies that a is to be squared, $(a+b)^2$ means that both a and b are to be squared, and to be connected with the sign $(+)$, so that $(a+b)^2 = a^2 + b^2$. This, however, is incorrect, as the learner will soon be able to see; the true meaning of $(a+b)^2$ being that the quantity $a+b$ is to be multiplied by itself, $a+b$, just as a^2 signifies a multiplied by a.

The following exercise is intended to impress on the memory the substance of the above remark.

1. Distinguish between $3a^2$ and $(3a)^2$; $a+b^2$ and $(a+b)^2$; $a+bc^2$, $a+(bc)^2$, and $(a+bc)^2$. Find the value of each of these quantities when $a=4$, $b=3$, $c=7$.

NOTE.—The horizontal line of the symbol $(\sqrt{})$ is, of course, nothing more than the vinculum, and its place may be supplied by brackets.

2. Write with brackets,

$$\sqrt{b-c},\quad \sqrt{3a+2b}-c,\quad \sqrt{3a+2b}-\overline{c-d}.$$

3. Explain the difference between $(a+b)\times c$ and $a+b\times c$; also between $a+b\times c+d$, $(a+b)\times c+d$, $a+b\times(c+d)$, and $(a+b)\times(c+d)$. Find the values of these quantities when $a=4$, $b=3$, $c=7$, $d=5$.

Is there any difference between

$$(a+b)\times(c+d),\quad (a+b)\,.\,(c+d),\quad \text{and}\quad (a+b)\,(c+d)\,?$$

4. What does $p+q\div m$ mean? What $(p+q)\div m$, $(p+q)\div m+n$, and $(p+q)\div(m+n)$? Find the values of these expressions when $p=12$, $q=8$, $m=4$, $n=1$. Write each of them, using the fractional form of denoting division instead of the sign $(\div)$.

5. Write algebraically the following:

(1.) Multiply $x+2$ by x, and add 3 to the result.

(2.) Multiply $x+2$ by $x+3$.

(3.) Multiply $a + b$ by $c + d + e$.

(4.) a plus the cube of b; the cube of a plus b; the cube of the whole quantity $a + b$.

(5.) Divide $1 + 2x$ by xy, and subtract 3 from the result.

(6.) Divide $1 + 2x$ by $xy - 3$.

(7.) To x add 2 multiplied by x.

(8.) To x add 2 multiplied by $x + 3$.

(9.) From 1 take x divided by y, and add 1 to the result.

(10.) From 1 take x divided by $y + 1$.

Exercise VI.

In the following questions, suppose $a = 5,\ b = 8,\ c = 4$.

1. Find the values of $a \backsim b,\ a - b,\ b.\frac{a}{c}$.

2. What is the *product* of a, b, and c, equal to? What the *quotient* of $5b$ by c?

3. Show that $7a + c < 5b$, and $> 10c - 2$; also that $2a - 3b + 4c = 2b - 14$.

4. Find the values of

$$a^2bc^3,\ \frac{3}{5}ac,\ 3ab + c^2,\ \frac{6a^2b}{5c} + \frac{5a - 2b,}{3},\ \frac{2}{3}(ac + 2b),$$

$$a + b\sqrt{a + 5b + c},\quad (a + b)\sqrt{a + 5\,b + c},\quad (b - a)^3,$$

$$\sqrt{(a + c)\,(a - c)},\ \sqrt{a + c} + \sqrt{a - c},\ \sqrt[3]{(b\,c - a)}.\ a\sqrt{\frac{bc}{a - 3}}.$$

5. Write algebraically, and then perform the operations indicated in, the following:

Multiply $a + b$ by $b + c$.

Divide $2a - c$ by 3, and multiply the result by $\frac{b}{c}$.

Multiply a by b, and a by c, and from the sum of the products take bc. Square the result.

6. Point out, by taking an example, the difference between the coefficient and the index of a letter.

ELEMENTARY RULES.

Exercise VII.

1. Define *like* and *unlike* quantities. Give instances of both.

2. Group together like quantities, with their proper signs, from the following:

(1.) $3a$, $2b$, ab, $5a^2b$, $-4b$, $3ba^2$, $-ab^2$, $2a$, $-4a^2b$, $3ba$.

(2.) $2a - 3b + c$, $a - 6c$, $-b + 3a$, $a - 5b - 3c$.

(3.) $a^3 - 3a^2b + 5ab^2$, $8a^3 + 6ab^2 - b^3 - 12a^2b$, $b^3 - 3b^2a + 3ba^2 - a^3$.

(4.) $3a - 12b + 5x^2 + 2xy - 7ac + 2x^2z + 11b - 3xy^2 - 8a + ac + 8y^2x - 3x^2 + 5x^2z$.

(5.) $x^3 + \frac{8}{9}x^2 - 7x + 3$, $x^3 + \frac{1}{2}x - 4 - \frac{3x^2}{4}$, $-\frac{x^3}{3} - x$.

3. State the rule for the addition of like quantities, and illustrate it by an example.

What is meant by the addition of unlike quantities?

4. Explain how several quantities, consisting of both like and unlike terms, are to be added together. Illustrate your rule by Example (3) in Question (2).

5. Add together

(1.) $x + y$ and $x + y$.
(4.) $x + y$ and $x - y + z$.
(2.) $x + y$ and $x - y$.
(5.) $-x + y$ and $x - y$.
(3.) $x - y$ and $x - y$.
(6.) $x - y + z$ and $y - z + x$.
(7.) $3y - 2x$ and $4x - 5y$.
(8.) $a + 2b - 3c$ and $2a - 5b + c$.
(9.) $2a - c + 3b$ and $-a + b + c$.
(10.) $2 + p - 3q$ and $3p + q - 1$.

(11.) $3m - 2n - 5p$ and $2n + 3p - 4q$.

(12.) $ab + 3c$ and $3ab - c$.

(13.) $2ax - 3by$ and $-5ax + 8by$.

(14.) $xy + 2x - 3y$ and $2xy - y$.

(15.) $2xy + 3xz - 4yz$ and $4xy - 3yz + 2xz$.

(16.) $x^2 + 2xy - 5y^2$ and $4x^2 - 5xy + y^2$.

(17.) $4c + 3x$, $2c + x$, and $5c - 4x$.

(18.) $2 - c$, $-5c + 1$, $-7 - 3c$, and $4 + c$.

(19.) $ax + x^2$, $-3ax + x^2$, and $-5ax - x^2$.

(20.) $2x - 3y + 4z$, $x + 4y - 6$, $-5x + 7z + 7$, and $4x - 2y + 3z - 10$.

(21.) $2a^2 + 2b^2 + 1$, $3ab + 3b^2 - 5$, $a^2 - ab + b^2$, $-3a^2 + 5ab + 7b^2 - 10$, and $-7a^2 - 6ab - 5b^2 - 20$.

(22.) $x^3 + ax^2 + a^2x + a^3$, $3a^3 - 6a^2x + 3ax^2$, $5x^3 - 6ax^2 + 7a^2x - 8a^3$, $4a^3 + 6ax^2$, and $-5a^2x - 7x^3$.

(23.) $\frac{2}{3}a + b$ and $3a - \frac{2}{5}b$.

(24.) $\frac{2}{3}a^2 + ab + \frac{3}{2}b^2$, $\frac{1}{4}a^2 - \frac{3}{5}ab + 4b^2$, and $3a^2 - 2ab - 7b^2$.

(25.) $\frac{x^4}{4} + z$, $y^4 - \frac{2}{3}z$, and $\frac{3x^4}{4} - 2y^4$.

(26.) $\frac{1}{3}x^2 - \frac{1}{4}y^2$, $\frac{5}{3}x^2 + \frac{1}{2}xy$, and $\frac{y^2}{8} - \frac{xy}{2}$.

Exercise VIII.

1. State the rules for subtracting one algebraical quantity from another.

Illustrate the several cases that can occur by examples.

2. (1.) From x take y. (3.) From $-x$ take y.
   (2.) From x take $-y$. (4.) From $-x$ take $-y$.
   (5.) From $x + y$ take $x - y$.
   (6.) From $x + y$ take $-x + y - z$.

(7.) From $5a + 4b + 8c$ take $3a + 2b + c$.
(8.) From $a - 2b + 3x$ take $3a - 5b - 5x$.
(9.) From $4a - 3b + 3c$ take $a + b + 10c - 20$.
(10.) From $6x^2 - 4xy + y^2$ take $5y^2 + 2x^2 + 3xy$.
(11.) From $3a - 4b + 5c + 1$ take $2a + 2b + 3d + 10$.
(12.) From $8ab - 5c + a$ take $2 - 3ab + 2c - 3b$.
(13.) From $3m - 2n - 5p$ take $2n - 3p + 4q$.
(14.) From
$a^3 + 3a^2b + 3ab^2 + b^3$ take $a^3 - 3a^2b + 3ab^2 - b^3$.
(15.) From
$x^3 - x^2y + 2xy - y^2$ take $3xy + 2y^2 - yx^2 + 1$.
(16.) From
$2abc - acx + bcx + 3abx$ take $4cax - 5bax + abc$.
(17.) From
$2x^3 - 4x^2y - 10y^3 + 6$ take $3x^3 + x^2y + xy^2 - 14y^3$.
(18.) From $\frac{2}{3}x^2 - 4x + \frac{3}{8}$ take $\frac{1}{3}x^2 + \frac{1}{2}x - \frac{1}{8}$.
(19.) From $2x^3 - \frac{4}{5}x^2y - 4$ take $\frac{x^3}{3} + \frac{1}{2} - \frac{3x^2y}{4}$.

3. Add the sum of two quantities to their difference. Take the difference from the sum.

Write at full length in words the propositions which may be deduced from your results.

4. The sum of two numbers is 54, and their difference is 36; find them.

5. One number exceeds another by 7, and their sum is 19; find the numbers.

6. A person leaves Shrewsbury by rail for Stafford, a distance of 30 miles, but falling asleep is carried on to Rugby, and finds on his return to Stafford that he has travelled 130 miles; how far is Rugby from Stafford?

7. Divide 15 shillings between 2 boys, so that one may have 3 shillings more than the other.

EXERCISE IX.

1. State the rules for multiplying

(1.) One single term by another.

(2.) A quantity consisting of several terms by a single term.

Illustrate your rules by examples.

2. Prove the general rule for multiplying powers of the same quantity together, viz. that

$$a^m \times a^n = a^{m+n}.$$

3. Multiply

(1.) ax by by.

(2.) $2ab$ by $-x$.

(3.) $-4x$ by $3y$.

(4.) $-8ab$ by $-cd$.

(5.) a^2 by ab.

(6.) $3a$ by $-2a^2$.

(7.) $-mn$ by mp.

(8.) $-x^2y$ by $-xy^2$.

(9.) ab^4 by a^2bc.

(10.) $2mx^3y$ by $-3nx^3y$.

(11.) $-10ab^2c^8$ by $-3ab^4c$.

(12.) $2a+b-3c$ by 2.

(13.) $a-4b$ by $3c$.

(14.) a^2-ab by ab.

(15.) $-ax+x^2$ by $-x^2$.

(16.) $x^2-2xy+y^2$ by xy.

(17.) $3x^3-2x^2y-y^3$ by $-4x^2y^2$.

(18.) a^m by a^n, and the result by a^p.

(19.) $2a^p$ by $-5a^q$.

Multiply together

(20.) ax^m, bx^n, and cx^p.

EXERCISE X.

1. State the rule for multiplying one quantity by another, when both consist of two or more terms.

Illustrate your rule by an example.

2. Multiply

(1.) $a+b$ by $c+d$.

(2.) $a+b$ by $c-d$.

(3.) $a+x$ by $b+x$.

(4.) $x-y$ by $x+y$.

(5.) $x + 3$ by $2x - 1$.
(8.) $ab - c$ by $a + bc$.
(6.) $-y + 2$ by $2 + y$.
(9.) $4x^2 + 3xy$ by $x - 3y$.
(7.) $3x - y$ by $2x + 5y$.
(10.) $1 - bc$ by $b + c$.
(11.) $a + b + c$ by $a + b$.
(12.) $-xy + x^2 + y^2$ by $x + y$.
(13.) $a - 2b - 3c$ by $-2a + b$.
(14.) $\frac{1}{2}x - 2y - \frac{1}{3}z$ by $6x - 12y$.
(15.) $\frac{1}{3}x - \frac{1}{4}y$ by $x + y$.
(16.) $x^2 + xy + y^2$ by $x - y$.
(17.) $1 - x + x^2 - x^3$ by $1 + x$.
(18.) $a^3 + a^2b + ab^2 + b^3$ by $a - b$.
(19.) $9x^2 - 6x + 3$ by $2x - 1$.
(20.) $a + b + c$ by $a + b - c$.
(21.) $a^2 + ax + x^2$ by $a^2 - ax + x^2$.
(22.) $x^2 + 2xy - 3y^2$ by $x^2 - 5xy + 4y^2$.
(23.) $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.

3. Multiply together
(1.) $x - 2$, $x + 3$, $x - 4$, and $x + 5$.
(2.) $2x - 1$, $x^2 + \frac{1}{4}$, and $2x + 1$.
(3.) $a - x$, $a^2 + x^2$, $a + x$.

EXERCISE XI.

1. Define the terms *Divisor*, *Dividend*, *Quotient*. State the connexion between them which results from the nature of division.

2. Explain how to divide one single algebraical term by another. Give an example.

State and prove the rule for dividing a quantity consisting of several terms by a single term.

3. Show that when m is greater than n, $a^m \div a^n = a^{m-n}$.

NOTE.—In the following examples every term of the proposed dividends contains the divisor exactly. When this is not the case, division can only be indicated by forming a fraction with the dividend for numerator, and divisor for denominator. The reduction of such fractions, when possible, will form the subject of a future Exercise.

4. Divide

(1.) $21a$ by 7.
(2.) $21a$ by a.
(3.) $3xy$ by y.
(4.) $3xy$ by $3x$.
(5.) $12abx$ by $3ab$.
(6.) $15acx^2$ by $-5ac$.
(7.) $-8bay$ by ab.
(8.) $-3x^2y^2$ by $-x^2$.
(9.) $75a^5b^2$ by $5a^2b^2$.
(10.) $6x^5y^3z^2$ by $-3x^2y$.
(11.) $-6x^5y^3z^2$ by xy^3.
(12.) $5x^2y^2$ by $\frac{1}{2}xy$.
(13.) $14h^2k^3$ by $-\frac{1}{3}hk$.
(14.) $\frac{4}{5}na^2c$ by $\frac{1}{5}n$.
(15.) $-\frac{6}{7}a^mb^2$ by $-\frac{2}{7}b$.
(16.) $27a^{25}$ by $9a^{16}$.

5. Divide

(1.) $6ab - 8ax$ by $2a$.
(2.) $21xy + 9x^2$ by $3x$.
(3.) $6abc + 12abx - 9a^2b$ by $3ab$.
(4.) $15a^2bc - 10acx^2$ by $-5ac$.
(5.) $24x^3y^2 - 15x^2y^2z + 6x^2y^3$ by $3x^2y$.
(6.) $a^2x^2 + 2abxy$ by $\frac{1}{2}ax$.

EXERCISE XII.

1. State the rule for dividing one quantity by another, when the divisor consists of two or more terms.

2. Write the numbers 276 and 23 algebraically, and divide the former by the latter according to your rule.

3. Divide

(1.) $x^2 + 7x + 6$ by $x + 1$.
(2.) $x^2 - 7x + 6$ by $x - 6$.
(3.) $x^2 + 4ax + 4a^2$ by $x + 2a$.
(4.) $x^2 - a^2$ by $x + a$.
(5.) $x^2 - a^2$ by $x - a$.
(6.) $6x^2 + 5xy - 4y^2$ by $3x + 4y$.
(7.) $ax - bx - ay + by$ by $a - b$.
(8.) $xy + 2y - 3x - 6$ by $x + 2$.
(9.) $9b^2 + 9ab + 2a^2$ by $a + 3b$.
(10.) $ab + ad - bc - cd$ by $b + d$.
(11.) $a^3 - 3a^2z + 3az^2 - z^3$ by $a - z$.
(12.) $48x^3 + 150a^3 - 96a^2x - 64ax^2$ by $2x - 3a$.
(13.) $x^3 - 40x - 63$ by $x - 7$.

4. Divide each of the quantities,

$$a^3 - x^3,\ a^4 - x^4,\ a^5 - x^5,\ a^6 - x^6, \text{ by } a - x.$$

Also, $a^3 + x^3$, $a^5 + x^5$, by $a + x$.

5. Divide

(1.) $4a^2m + 8amb - 12amc$ by $a + 2b - 3c$.
(2.) $x^3 - 5x^2 + 10x - 8$ by $x^2 - 3x + 4$.
(3.) $x^6 - 2a^3x^3 + a^6$ by $x^2 - 2ax + a^2$.
(4.) $1 - 6x^5 + 5x^6$ by $1 - 2x + x^2$.
(5.) $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

6. Divide

(1.) $p^2 + pq + 2pr - 2q^2 + 7qr - 3r^2$ by $p - q + 2r$.
(2.) $81a^4 - 16b^4$ by $3a - 2b$.
(3.) a by $a - x$ to 6 terms.

7. What is the remainder after dividing $1 + 2x$ by $1 - 3x$ to 5 terms? Prove the correctness of your result.

8. The dividend and quotient being $x^3 - a^3$, and $x^2 + ax + a^2$ respectively; find the divisor.

Exercise XIII.

1. Add together

(1.) $3a - 2b + 4c$, $9b - 3c$, and $8a^2 + 2c - b$.

(2.) $x^2 + 3xy - y^2$, $4z^2 - 4xz + x^2$, and $3y^2 - 2yz - 7z^2$.

2. Find the sum of

$$3a^2b^2c + 2ab^4 - \frac{1}{4}c^4,\ a^2b^2c - \frac{3}{4}c^4, \text{ and } \frac{2}{3}ab^4 - 2.$$

3. Subtract

(1.) $3x - 4y + z$ from $5x + 3y - 2z$.

(2.) $x^2 + 2xy - 3y^2$ from $-x^2 - 5xy + 4y^2$.

(3.) $3ab - c^2 + ac$ from $2c^2 - 5 - 4ac - a^2$.

(4.) $\frac{1}{2} - \frac{3}{5}x - x^2$ from $2 - \frac{3}{2}x^2$.

4. Multiply

(1.) $6a^2 + 7ax - 3x^2$ by $5ax$.

(2.) $x - y + z$ by $x + y - z$.

(3.) $x^4 - 3x^3 + 2x^2 - 5$ by $x - 2$.

(4.) $9x^2 + 3xy + y^2 - 6x + 2y - 4$ by $3x - y + 2$.

(5.) $x^m - x^{m-3} + x^3 - 1$ by $x^3 + 1$.

5. Multiply together

$$x - 1,\ x - 2,\ x - 3,\ x^2 - 1.$$

6. Divide a^3b^5c by a^2bc; $-6mnx^4y^6$ by $2mx^3y$.

7. Divide

(1.) $6x^2 + 13xy - 5y^2$ by $2x + 5y$.

(2.) $a^4 + a^2x^2 + x^4$ by $a^2 - ax + x^2$.

(3.) $x^3 - y^3 + 125z^3 + 15xyz$ by $x - y + 5z$.

(4.) $x^{2n+1} - x^{2n} - x^{n+1} + x^n + x - 1$ by $x^{2n} - x^n + 1$.

8. Find the product of $x^2 - a^2$, multiplied by the quotient of $x^5 - a^5$ by $x - a$.

BRACKETS.

Exercise XIV.

Observation.—It has been stated that quantities enclosed within brackets must be regarded as *wholes*. Conversely, when we wish to treat a quantity as *a whole*, it should be enclosed in brackets. Questions of addition and subtraction may thus be indicated algebraically by aid of the signs (+), (—), and brackets, bearing in mind that the strict meaning of these signs is, that what follows the sign *is to be* added to, or subtracted from, what precedes. Thus, the questions, "Add together $3c + 4x$, $2c - x$, and $-7c + 3x$," and "Subtract $2a - b + 3c$ from $3a - b - 6c$," may be translated algebraically as follows:

$$(3c + 4x) + (2c - x) + (-7c + 3x). \qquad (\alpha.)$$

$$(3a - b - 6c) - (2a - b + 3c). \qquad (\beta.)$$

Such expressions will be constantly presenting themselves in the treatment of fractions and equations; and the learner will find it convenient to replace the ordinary rules for performing the additions and subtractions indicated (viz. by arranging the proposed quantities under one another, like terms under like terms, &c.) by what may be termed *horizontal* addition and subtraction. Thus, let us perform the operations indicated in (α), as though no two terms were *like*, that is, by connecting all the terms with their proper signs; this gives us

$$3c + 4x + 2c - x - 7c + 3x; \qquad (\gamma.)$$

Adding horizontally the c's and x's, this expression is reduced to

$$-2c + 6x.$$

Again, to perform the operations indicated in (β), annex every term of the second quantity within brackets with its sign changed to the first quantity, as in the subtraction of unlike quantities; we get as our result,

$$3a - b - 6c - {}^{*}2a + b - 3c; \qquad (\delta.)$$

Adding horizontally as before, the final result is

$$a - 9c.$$

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* The term $2a$, as it stands within the bracket, has no sign; its proper sign is therefore +, and this, when the subtraction is performed, becomes −.

1. From a comparison of the above equalities, (α), (γ); (β), (δ), deduce the rules for removing brackets preceded by the signs (+) and (–).

2. Simplify the following expressions:—

$$2a - 3b + c + a - 6c - b + 3a + a - 5b - 3c.$$
$$3ax - x^2 + a^2 + 4x^2 - 2ax - 3 + 2a^2.$$

3. Indicate by brackets, and perform the additions required in Ex. VII. 5.

4. Indicate by brackets, and perform the subtractions required in Ex. VIII. 2.

Exercise XV.

Observation.—The student will have learnt from Ex. V. the office of brackets in indicating the operations of Multiplication, Division, &c. The usual difficulties presented by such Exercises as the following will be avoided, by keeping in mind that *a bracket must never be removed till the operations indicated, (sometimes more than one,) have been performed.*

Simplify the following expressions:—

(1.) $x\ (a + y)\ - ax.$
(2.) $2a\ (a + x)\ + 2a\ (a - x).$
(3.) $2a\ (a + x)\ - 2a\ (a - x).$
(4.) $8\ (1 - y)\ - 3\ (2 + 3y).$
(5.) $(x + y)\ (x - y)\ + y^2.$
(6.) $(x + y)^2\ +\ (x - y)^2.$
(7.) $(a + b)^2\ -\ (a - b)^2.$
(8.) $3\ (x - y)\ + 4 \times (y - z).$
(9.) $a -\ (a - \overline{b + c}).$
(10.) $ax - a\ \{x -\ (y - z)\}.$
(11.) $3\ \{1 + x\ (1 - x)\} + 3\ \{1 - x\ (1 + x)\}.$
(12.) $(a + b)\ (x + y)\ +\ (a - b)\ (x - y).$
(13.) $(a + b)\ (x - y)\ -\ (a - b)\ (x + y).$
(14.) $\{x^2 -\ (x - y)^2\}\ y\ (2x + y)\ + y^4.$

MEASURES AND MULTIPLES.

EXERCISE XVI.

1. DEFINE the term "*measure*," and explain your definition by an example.

The fraction $\frac{2}{3}$ is contained exactly 6 times in 4; is $\frac{2}{3}$ called a "measure" of 4?

2. What is meant by a "*common measure?*" What are the common measures of 6, and 9; of 16, and 20; of 8, 12, and 20? What is the greatest common measure of 18, and 24; of 48, and 56; of 6, 18, and 24?

3. Distinguish between *prime* and *composite* numbers. When are two composite numbers said to be *prime to each other?*

4. Explain how to resolve a composite number into its simple factors.

5. Resolve into simple factors the numbers 108, 168, 420, 462, 819.

What are the common measures, and greatest common measure, of the first two of these numbers? of the last three?

6. Which of the numbers 56, 131, 157, 135, are composite? which prime? Are the composite numbers prime to each other?

7. Write down all the common measures of $4a^2bc$, and $6abc^2$. What is their greatest common measure?

8. Find the quantities of which the simple component factors are a, $x - a$; $a + b$, $a - b$; $a + b$, $a + b$; $a - b$, $a - b$; $x + y$, $x^2 - xy + y^2$; $x - y$, $x^2 + xy + y^2$.

9. What are the simple factors of the following quantities?

$$2ax - x^2, \quad x^2 + 2xy + y^2, \quad x^2 - 2xy + y^2, \quad x^2 - y^2, \quad x^3 + y^3, \quad x^3 - y^3.$$

NOTE.—The results of Questions (8) and (9) should be carefully committed to memory.

10. Find the greatest common measure of

(1.) 264 and 110.
(2.) 240, 42, and 78.
(3.) ab and a^2.
(4.) abc and bcx.
(5.) a^2xy and ay^3.
(6.) $6mxy^3$ and $8mnx^2yz$.
(7.) $21a^4x^3$ and $9bx^5y^2$.
(8.) abx, acx, and bcx.
(9.) $5a^2bx$, $3abx^2$, and ab^2x.
(10.) x^2y^2, $9xy^3$, and $24mx^5$.
(11.) $ax - a^2$, and ax.
(12.) $ax - a^2$, and $ax + a^2$.
(13.) $a^2 - x^2$, and $a^2 + 2ax + x^2$
(14.) $a^2 - x^2$, and $a^2 - 2ax + x^2$.
(15.) $x^3 + y^3$, and $x^2 + 2xy + y^2$.
(16.) $x^3 - y^3$, and $x^2 - y^2$.

EXERCISE XVII.

1. Define the terms "*multiple*," "*common multiple*," "*least common multiple*." Illustrate by an arithmetical, and also by an algebraical, example.

2. Explain how to find the least common multiple of two or more simple quantities. Ex. Of 6, 9, 20; of $3a$, $6ax$, $12a^2x$.

3. Find the least common multiple of

(1.) 12 and 20.
(2.) 9, 15, and 21.
(3.) 8, 7, 14, and 24.
(4.) 7, 18, 28, 30, and 42.

(5.) 15, 16, 18, 20, 24, and 25.
(6.) mx and nx.
(7.) $3ax$ and $2bx$.
(8.) $5abc$ and $3a^2x$.
(9.) x^2, $2x$, and $3x$.
(10.) $2xy$, $3xz$, and $6yz$.
(11.) $4a^2b$, $2ax^2$, and $14b^2x$.
(12.) $3x^3y$, $5x^2y^2$, and $4xy^3$.
(13.) $6a^2$, $9ax^3$, and $24x^5$.
(14.) $5abc$, $7acd$, and $14bcd$.
(15.) $4a^3cx$, $6a^2bcx$, and $36acx^3$.
(16.) $2x$, and $x(x-y)$.
(17.) $(x+y)(x-y)$, and $3(x+y)$.
(18.) $4(x^2-y^2)$, and $x-y$.
(19.) $x^2+2xy+y^2$, and $5(x+y)$.
(20.) $(x+y)^2$, and x^2-y^2.

FRACTIONS.

EXERCISE XVIII.

1. WRITE the fraction whose *numerator* is p, and *denominator* q. What is the meaning of this fraction?

2. Show that the fraction $\frac{a}{b}$ represents the bth part of a.

3. Show that the value of a fraction is not altered by
   (1.) multiplying numerator and denominator by the same quantity.
   (2.) dividing numerator and denominator by the same quantity.

4. What is meant by reducing fractions to their lowest terms, and how is the simplification done?

5. Reduce to lowest terms the following fractions:—

(1.) $\frac{mx}{nx}$.

(2.) $\frac{6abx}{4b}$.

(3.) $\frac{15ab^3}{5abc}$.

(4.) $\frac{3x^2yz}{9xyz^2}$.

(5.) $\frac{14ab^2x^3}{8a^3bx^2}$.

(6.) $\frac{40ax^5y}{5a^2x^2}$.

(7.) $\frac{-5a^2xy}{10axz}$.

(8.) $\frac{8a^3x^4z}{-12ax^5y^2}$.

(9.) $\frac{-30ab^2c^3x^4}{-45a^4b^3c^2x}$.

(10.) $\frac{ax+bx}{2xy}$.

(11.) $\frac{y^2-2y}{ay^2}$.

(12.) $\frac{14a^3b-7a^2b^2}{7ab}$.

(13.) $\frac{15ab^3-3a^2bc^2}{5abc}$.

(14.) $\frac{3x^2-2ax}{2x^2-3ax}$.

(15.) $\frac{12a^2-6ab}{9ax-3ay}$.

(16.) $\frac{ax+bx+cx}{ax-bx+cx}$.

(17.) $\frac{a(x-y)}{x^2-y^2}$.

(18.) $\frac{x^2+2xy+y^2}{x^3+y^3}$.

(19.) $\frac{(x+y)(x^2+xy+y^2)}{x^3-y^3}$.

6. Change the fractions

$$\frac{3x}{2},\ \frac{7bx}{a^2},\ \frac{3a^2x}{5b^2y},\ \frac{2a^2z}{-3xy^2},\ \frac{m-n}{x},\ \frac{x-4}{3a-b},$$

into their equivalents with denominators respectively,

$$4b,\ a^3bx^2,\ 15a^2b^2y^2,\ 9ax^4y^2,\ 3xy,\ 3ax-bx.$$

Exercise XIX.

1. State and explain the rules for the addition and subtraction of Algebraic fractions.

2. Add together

(1.) $\frac{3a}{7}, \frac{4a}{7}, \frac{9a}{7}$.

(2.) $\frac{x}{4}, \frac{x-3}{4}, \frac{x+1}{4}$.

(3.) $\frac{ax}{5}, \frac{-2ax}{5}, \frac{3ax}{5}$.

(4.) $\frac{xy}{4}, \frac{xy}{8}$.

(5.) $\frac{2ax}{3}, \frac{5ax}{4}, \frac{ax}{12}$.

(6.) $\frac{a+b}{3}, \frac{a-b}{9}$.

(7.) $\frac{x}{3}, \frac{x}{5}, \frac{x}{9}$.

(8.) $\frac{3}{x}, \frac{5}{x}, \frac{9}{x}$.

(9.) $\frac{a}{xy}, \frac{3a}{xy}, \frac{2a}{xy}$.

(10.) $\frac{x}{a}, \frac{3x}{4}$.

(11.) $\frac{b}{x}, \frac{2b}{xy}, \frac{3b}{x^2}$.

(12.) $\frac{1}{xy}, \frac{2}{xz}, \frac{3}{yz}$.

(13.) $\frac{3}{4x}, \frac{4}{3x}, \frac{1}{6x}, \frac{5}{12x}$.

(14.) $\frac{a}{x}, \frac{b}{y}, \frac{c}{z}$.

(15.) $\frac{m}{a^2b}, \frac{n}{ab^2}, \frac{p}{a^2b^2}$.

(16.) $\frac{x+1}{2}, \frac{x+2}{3}$.

(17.) $\frac{x}{11}, \frac{3x-5}{7}$.

(18.) $\frac{3x+4}{5}, \frac{7x-3}{2}$.

(19.) $\frac{4x}{x-a}, \frac{3x}{x-a}$.

(20.) $\frac{3x}{2(x+a)}, \frac{x}{3(x+a)}$.

(21.) $\frac{1}{x-1}, \frac{1}{x+1}$.

(22.) $\frac{a}{a-x}, \frac{a}{a+x}$.

(23.) $x, \frac{x^2-1}{x+1}$.

(24.) $5x, \frac{7x+9}{4x+3}$.

(25.) $\frac{x}{3}, \frac{x+8}{4x+10}$.

(26.) $\frac{5x-4}{x-1}, \frac{8x-30}{2x-7}$.

(27.) $\frac{3x-3y}{x^2+2xy+y^2}, \frac{3}{x+y}$.

(28.) $\frac{2}{x+y}, \frac{2y}{x^2-y^2}$.

Exercise XX.

Observation.—The question 'Subtract $\frac{3x-1}{4}$ from $\frac{5x-3}{6}$' is written algebraically

$$\frac{5x-3}{6} - \frac{3x-1}{4},$$

the line separating numerator and denominator of the latter fraction serving as a vinculum; so that virtually this fraction is in a bracket: hence, when the fractions have been brought to a common denominator 12, the numerators should be combined as follows,

$$\frac{10x-6-(9x-3)}{12};$$

whence, removing the bracket, we get $\frac{10x-6-9x+3}{12}$, or $\frac{x-3}{12}$, for result. The omission of the bracket is one of the commonest sources of error with beginners. Of course it may be dispensed with if the requisite alterations in the signs be at once made; but it will be safer for the learner, at first, to insert all the steps.

1. Subtract

(1.) $\frac{2x}{7}$ from $\frac{5x}{7}$.

(2.) $\frac{x}{3}$ from $\frac{x+a}{3}$.

(3.) $\frac{3a}{4}$ from $\frac{5a}{6}$.

(4.) $\frac{3x}{5}$ from $2x$.

(5.) $\frac{x-2}{7}$ from $\frac{4x-1}{21}$.

(6.) $\frac{3x-5}{7}$ from $\frac{x-7}{11}$.

(7.) $\frac{x+5}{x-5}$ from $\frac{6x+13}{3(x-5)}$.

(8.) $\frac{x}{1-x}$ from $\frac{x^2+1}{x}$.

(9.) $\frac{2x}{x+1}$ from $\frac{3x}{x+2}$.

(10.) $\frac{a}{x+a}$ from $\frac{a}{x-a}$.

(11.) $\frac{a-x}{a+x}$ from $\frac{a+x}{a-x}$.

(12.) $\frac{x-y}{x+y}$ from $\frac{2x^2-2xy}{x^2+2xy+y^2}$.

2. Simplify the following:

(1.) $\frac{x}{2} - \frac{x}{3} + \frac{x}{4}$.

(2.) $x - \frac{x^2}{x+1}$.

(3.) $x - \frac{x^2}{x+1} + \frac{x}{x-1}$.

(4.) $\frac{1}{1-x} - \frac{1}{2(1+x)} - \frac{2}{x}$.

(5.) $\frac{1}{2(x-1)} - \frac{4}{x-2}$.

(6.) $\frac{x}{x-3} - \frac{x-3}{x} + \frac{x}{x+3} - \frac{x+3}{x}$.

(7.) $\frac{1}{x^3} + \frac{1}{x^2} - \frac{1}{x}$.

(8.) $\frac{x+1}{2} - \frac{x+2}{3} + \frac{5x+1}{4}$.

(9.) $\frac{x-7}{11} + 2x - \frac{3x-5}{7}$.

(10.) $\frac{x}{9} - \frac{x^2}{2x-3} + \frac{x}{3(2x-3)}$.

(11.) $\frac{6x+13}{15} - \frac{3x+5}{5x-25} - \frac{2x}{5}$.

(12.) $\frac{75-x}{3(x+1)} + \frac{80x+9}{3x+2} - 5 - \frac{23}{x+1}$.

(13.) $\frac{x-1}{x+1} + \frac{x+3}{x-3} - \frac{2x+4}{x-2}$.

(14.) $x - 9 + \frac{x-1}{6} - \frac{3x}{x+2}$.

EXERCISE XXI.

1. State and explain the Rules for

(1.) the multiplication,
(2.) the division,

of a fraction by a whole number, or algebraical quantity not in the form of a fraction.

How may the results of such multiplications and divisions be obtained most easily in their lowest terms?

2. Multiply

(1.) $\frac{x}{3}$ by 5.

(2.) $\frac{3x}{4}$ by 4.

(3.) $\frac{a}{6}$ by 3.

(4.) $\frac{3a}{7}$ by 14.

(5.) $\frac{x+y}{5}$ by 20.

(6.) $\frac{x-y}{20}$ by 5.

(7.) $\frac{2x-3}{}$ by 14.

(8.) $\frac{x-7}{15}$ by 8.

(9.) $\frac{3x-5}{4\frac{1}{2}}$ by 9.

(10.) $\frac{a}{b}$ by x.

(11.) $\frac{x}{y}$ by y.

(12.) $\frac{3x}{5y}$ by $10xy$.

(13.) $\frac{x}{3y^2}$ by $2xy^2$.

(14.) $\frac{ma}{4nb}$ by $8na$.

(15.) $\frac{5y}{9x^2z}$ by $4ay$.

(16.) $\frac{3abx}{5c^2yz}$ by $-c^3y^2$.

3. Divide

(1.) $\frac{7x}{5}$ by 7.

(2.) $\frac{15a}{8b}$ by 5.

(3.) $\frac{4x}{9}$ by 3.

(4.) $\frac{6xy}{7}$ by 4.

(5.) $\frac{2ab}{3}$ by a.

(6.) $\frac{10x^2}{13y}$ by $5x$.

(7.) $\frac{2x}{3}$ by $7y$.

(8.) $\frac{4abc}{5}$ by $8bcx$.

(9.) $\frac{x^2+xy}{7y}$ by x.

(10.) $\frac{2ab-6ac}{bc}$ by $2a$.

(11.) $\frac{3a-x}{y}$ by $4xy$.

(12.) $\frac{a^2-x^2}{ax}$ by $a-x$.

(13.) $\frac{(x+y)^2}{x-y}$ by $x+y$.

(14.) $\frac{x}{x+y}$ by $x-y$.

4. Perform the operations indicated in the following expressions:

$$3\left(\frac{7+x}{5}\right),\ 2\cdot\frac{2x-y}{3},\ 8\left(\frac{5y-7}{12}\right),\ 3\cdot\frac{4x-3}{21x};$$

$$5x\left(\frac{3a-2x}{x^2}\right),\ (a+x)\left(\frac{x}{2\,(a+x)}\right),\ \overline{a-x}\cdot\frac{a}{a^2-x^2},\ (a^2-x)\cdot\frac{a}{a-x}.$$

Exercise XXII.

1. State the rules for

(1.) multiplying one fraction by another,
(2.) dividing one fraction by another.

Show how the rule for division follows from that for multiplication.

2. Multiply

(1.) $\frac{x}{3}$ by $\frac{4}{5}$.

(2.) $\frac{6y}{5x}$ by $\frac{2x}{3}$.

(3.) $\frac{13\,(x^2-1)}{7x}$ by $\frac{7x^2}{13}$.

(4.) $\frac{6\,(x^2+xy)}{35}$ by $\frac{7y}{3x}$.

(5.) $\frac{x+y}{x}$ by $\frac{x-y}{y}$.

(6.) $\frac{x^2-y^2}{xy}$ by $\frac{x}{x+y}$.

(7.) $\frac{x+1}{x+2}$ by $\frac{x+3}{x+4}$.

(8.) $\frac{x-y}{x+y}$ by $\frac{x^2+xy+y^2}{x^2-xy+y^2}$.

(9.) $\frac{x^3-y^3}{x^3+y^3}$ by $\frac{x+y}{x-y}$.

Divide

(1.) $\frac{2a}{3b}$ by $\frac{4}{5}$.

(2.) $\frac{3x^2}{7y}$ by $\frac{9x}{14y^2}$.

(3.) $\frac{a^2-ab}{10b}$ by $\frac{4a}{5x}$.

(4.) $\frac{a-b}{a}$ by $\frac{a+b}{a}$.

(5.) $\frac{3x^2-6xy}{5y^2z}$ by $\frac{3x}{y^2}$.

(6.) $\frac{(x+y)^2}{x^2}$ by $\frac{x+y}{y}$.

(7.) $\dfrac{x^2-y^2}{xy}$ by $\dfrac{x-y}{ay}$. (8.) $\dfrac{x^3-y^3}{x^3+y^3}$ by $\dfrac{x-y}{x+y}$.

(9.) $\dfrac{2x-3}{4}$ by $\dfrac{x}{x-1}$.

3. Multiply $x+\dfrac{1}{x}$ by $x-\dfrac{1}{x}$; $\dfrac{1}{a}+\dfrac{1}{b}$ by $a+b$.

4. Divide $\dfrac{1}{a}\left(1+\dfrac{1}{b^3}\right)$ by $\dfrac{b+1}{ab^2}$; $a+b+\dfrac{b^2}{a}$ by $a+b+\dfrac{a^2}{b}$.

Exercise XXIII.

Reduce to their simplest forms the following expressions:

(1.) $\dfrac{x+\dfrac{1}{x}}{x^2+1}$.

(2.) $\left(\dfrac{2x+1}{x-2}\right)\left(1-\dfrac{2}{x}\right)$.

(3.) $\left(1-\dfrac{x}{4-x}\right)\left(1-\dfrac{x}{4}\right)$.

(4.) $\dfrac{x^4-a^4}{(x-a)^2}\div\left(x+\dfrac{a^2}{x}\right)$.

(5.) $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}$.

(6.) $\left(a-\dfrac{x^2}{a}\right)\div\dfrac{1}{\dfrac{a}{x}+\dfrac{x}{a}}$.

(7.) $\dfrac{1-x}{1+x}\cdot\left(1+\dfrac{1-x^2}{1+x^2}\right)$

(8.) $\left(\dfrac{1}{1+x}+\dfrac{x}{1-x}\right)\div\left(\dfrac{1}{1-x}+\dfrac{x}{1+x}\right)$.

(9.) $\dfrac{\dfrac{a}{1+a}+\dfrac{1-a}{a}}{\dfrac{a}{1+a}-\dfrac{1-a}{a}}$.

(10.) $\dfrac{a+\dfrac{b-a}{1+ba}}{1-a\,.\,\dfrac{b-a}{1-ba}}$.

(11.) $\left(\frac{1}{1+x}+\frac{1}{1-x}\right)\div\frac{1}{1-x^2}$.

(12.) $\dfrac{1}{x-\dfrac{2xy}{x+y}}$.

(13.) $\left\{1\div\left(x-\frac{2xy}{x+y}\right)\right\}\times\left(\frac{x-y}{x+y}\right)$.

(14.) $1\div\left\{\left(x-\frac{2xy}{x+y}\right)\times\left(\frac{x-y}{x+y}\right)\right\}$.

(15.) $\dfrac{a+\dfrac{2a}{a-3}}{a-\dfrac{2a}{a-3}}\times\dfrac{1}{a^2-1}$.

(16.) $\dfrac{1}{\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x-2}}$.

(17.) $\frac{2}{3}(x-1)x\{x+1-\frac{1}{2}(2x-1)\}$.

(18.) $\dfrac{1}{\dfrac{1}{x+\dfrac{1}{x}}}$.

SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

EXERCISE XXIV.

1. WHAT is meant by a *Simple Equation?* Distinguish between an *identity* and an *equation.* Explain the general method of proceeding in the solution of Simple Equations.

How may the correctness of a solution be tested?

2. Solve the Equations:

(1.) $17x + 12 = 54 - 4x.$

(2.) $6x - 16 = 5x - 9.$

(3.) $4x - 4 = 3x + 2.$

(4.) $13x - 25 = 9x - 13.$

(5.) $9x + 2 = 7x + 15.$

(6.) $3x + 39 = 2x + 42.$

(7.) $5x - 3x = 36 - x.$

(8.) $7x - 12 + 5 = 8x - 24.$

(9.) $30 - x + 12 = 3x + 2.$

(10.) $6x - 30 = 60 - 8x - 20.$

(11.) $9x - 16 = 3x - (x - 5).$

(12.) $5x - 10 + 2x = 6(x + 1) - x.$

(13.) $2(5x + 1) = x + 2.$

(14.) $5(x + 1) = 2(x - 5) + 8.$

(15.) $3(x - 2) - 2(x - 3) - 5 = 0.$

(16.) $(3 - 2x) + 3(1 - 4x) - 4(1 - 5x) = 2(1 - 3x.)$

(17.) $3x - \frac{1}{3} = 44\frac{2}{3} - 2x.$

(18.) $2x - 3 = \frac{1}{5} - 6x.$

(19.) $5x + 3 = \frac{29}{3} - \frac{15x}{9}.$

(20.) $30x - 3\frac{1}{2} = 3\frac{1}{2} + 2x.$

(21.) $3x - \frac{x}{3} = 42 - 2x.$

(22.) $3x + \frac{5}{4}x = 34.$

(23.) $x - \frac{x}{2} + \frac{x}{3} = 5.$

(24.) $\frac{x}{2} - \frac{x}{3} = \frac{x}{4} - 1.$

(25.) $\frac{1}{3}x + \frac{1}{4}x = 19 - x.$

(26.) $\frac{x}{2} - \frac{x}{3} = \frac{x}{4} - \frac{x}{5}.$

(27.) $x + \frac{x}{2} - \frac{x}{3} = 17 - \frac{x}{4}.$

(28.) $15x - 9 = 20 + \frac{x + 1}{3}.$

(29.) $\frac{x}{2} + \frac{x + 1}{3} = 14 + \frac{6 - x}{4}.$

(30.) $\frac{x}{3} = \frac{5x - 9}{4} - \frac{x - 1}{2}.$

(31.) $\dfrac{5x-12}{3} - \dfrac{3x-5}{7} = \dfrac{x-6}{4}$.

(32.) $\dfrac{x}{4} - \dfrac{10x+4}{3} = \dfrac{8x-9}{3} - \dfrac{3x}{4}$.

(33.) $\dfrac{x-6}{11} - \dfrac{3x-2}{7} + \dfrac{125}{77} = 2x - 15$.

(34.) $\dfrac{x+5}{2} - \dfrac{9-x}{5} = \dfrac{3x+2}{20} + 3\frac{1}{5}$.

(35.) $x - \dfrac{x-4}{3} = 7\frac{3}{4} - \dfrac{x+8}{5} + \dfrac{x-2}{4}$.

(36.) $\dfrac{9x-2}{2} - \left(x - \dfrac{x+4}{7}\right) = 36$.

(37.) $\dfrac{3x}{14} - \dfrac{2x-7}{21} + 3\frac{1}{4} = \dfrac{x-4}{4}$.

(38.) $\dfrac{18x+1}{29} - \dfrac{402-27x}{12} = 9 - \dfrac{471-54x}{2}$.

(39.) $\dfrac{\frac{3}{2}x - 2}{8} + 19 = x - \dfrac{(x+1) - \frac{1}{5}(2x-4) + x}{11}$.

(40.) $\dfrac{5-3x}{5} - \dfrac{2x+9}{1\frac{1}{2}} + \dfrac{11x+15}{3} = 0$.

(41.) $\cdot 3x + \cdot 4 - 1\cdot 75x + \cdot 375 = \cdot 125x - 1$.

(42.) $\dfrac{6}{x} + \dfrac{2}{x} = 4$.

(43.) $\dfrac{8}{3x} + \dfrac{1}{9} = \dfrac{3}{x}$.

(44.) $\dfrac{1}{x} - \dfrac{1}{3x} = \dfrac{7}{3} - \dfrac{1}{2x}$.

(45.) $\dfrac{3x+13}{15} - \dfrac{3x+10}{10(x-5)} = \dfrac{x}{5}$.

(46.) $\dfrac{4x-5}{9}+\dfrac{7x-43}{5x-22}=\dfrac{8x+3}{18}$.

(47.) $\dfrac{6-35x}{105}-\dfrac{5-4x-2x^2}{14x}=\dfrac{4+3x}{21}-\dfrac{2x-\dfrac{1}{5}}{6}$.

(48.) $\dfrac{6x-7\frac{1}{3}}{13-2x}+2x+\dfrac{1+16x}{24}=4\frac{5}{12}-\dfrac{12\frac{5}{8}-8x}{3}$.

(49.) $\dfrac{25-\dfrac{1}{3}(x+1)}{x+2}+\dfrac{16x+20\frac{1}{5}}{3x+5}=5+\dfrac{23}{x+2}$.

(50.) $\dfrac{2x+1}{2x+3}-\dfrac{1-2x}{3-2x}=1-\dfrac{4x^2+3x-4}{4x^2-9}$.

EXERCISE XXV.

OBSERVATION.—The following Equations, in which the value of the unknown quantity is to be obtained in terms of *letters*, are termed *Literal* Equations in contradistinction to those of the last Exercise, which are called *Numerical* Equations.

The solution of the equation $ax+bx=c$, where a, b, c, stand for known quantities, consists in determining x in terms of a, b, and c; so that if special numerical values were assigned to these letters, the corresponding numerical value of x might be found at once. The learner has hitherto considered ax and bx as unlike quantities, but regarding them as *literal* multiples of x we may proceed a step further in adding them together than merely connecting them with the sign $(+)$. Thus, just as $3x+2x=(3+2)x$, or $5x$, so, collecting coefficients, (or, what is the same thing, resolving into simple factors,) $ax+bx=(a+b)x$; and if this quantity, as in the proposed equation, be equal to c, we at once obtain by dividing by the compound coefficient of x,

$$x=\frac{c}{a+b}.$$

The following is a more complicated example.

Given $\qquad \dfrac{a^2x}{b-c}-dc=bx-ac$; find x.

Transposing, $\frac{a^2x}{b-c} - bx = dc - ac = (d - a)\,c.$

Collecting coefficients,

$$\left(\frac{a^2}{b-c} - b\right)x = (d-a)\,c.$$

Simplifying the coefficient of x,

$$\frac{a^2 - b^2 + bc}{b-c} \,.\, x = (d-a)\,c.$$

whence,
$$x = \frac{c\,(d-a)\,(b-c)}{a^2 - b^2 + bc}\,.$$

Solve the following Equations:

(1.) $ax = bx + c.$

(2.) $ax - bx = m + n.$

(3.) $mx - x - n = p.$

(4.) $ax + b^2 = a^2 - bx.$

(5.) $\frac{2a}{x} = \frac{5}{x} + 4.$

(6.) $\frac{x}{a} + \frac{x}{b} = c.$

(7.) $\frac{a}{bx} + \frac{b}{ax} = a^2 + b^2.$

(8.) $bx + 3x - a = 7x + 2c.$

(9.) $\frac{5}{6}(x - 3a) - \frac{1}{5}(2x - 3b) = \frac{54}{5}a + 11b.$

(10.) $\frac{3x - a}{b} + \frac{x + 2b}{c} = \frac{7x}{c} - \frac{a}{4}$

(11.) $(a + x)\,(b + x) - a\,(b + c) = \frac{a^2c}{b} + x^2.$

(12.) $ax^2 + a^3 = (ax + b^2)\,(a + x).$

(13.) $\frac{3ac}{a + b} + \frac{a^2b}{(a + b)^2} + \frac{(2a + b)\,bx}{a\,(a + b)^2} = \frac{3cx}{b} + \frac{x}{a}\,.$

PROBLEMS PRODUCING SIMPLE EQUATIONS OF ONE UNKNOWN QUANTITY.

Exercise XXVI.

1. What number is that, to the double of which if 16 be added, the sum will be 90?

2. What number is that, the treble of which exceeds its half by 40?

3. Two workmen were employed to build a wall 120 feet long; one did 5 yards, and the other 7, per day. In what time did they finish it?

4. Find that number, to which if its third part be added, the sum will equal four times the number diminished by 8.

5. A mercer bought 3 pieces of silk which together measured 48 yards; the second was twice, and the third three times as long as the first. What were the respective lengths of the pieces?

6. A farmer sold 11 bushels of barley at a certain price; and afterwards 19 bushels at the same rate; and at the second time received 68 shillings more than at the first. What was the price of a bushel?

7. A person bought 162 gallons of beer, which exactly filled 4 casks; the first held nine times as much as the fourth, the second three times as much as the third, and the third twice as much as the fourth. How many gallons did each hold?

8. Three pieces of iron together weigh 37 pounds. The second weighs 4 pounds more than the first, and 5 pounds less than the third. What are their respective weights?

9. A vintner fills a cask containing 106 gallons, with a mixture of brandy, wine, and water. There are 23 gallons more of water than of brandy, and 9 more of wine than of water. How many are there of each?

10. A gentleman buys 4 horses; for the second of which he gives 15*l.* more than for the first; for the third 9*l.* more than for the second; and for the fourth 8*l.* more than for the third. The sum paid for all was 227*l.* How much does each cost?

11. A poor man had 5 children, the eldest of which could earn 9*d.* a week more than the second; the second 8*d.* more than the third; the third 5*d.* more than the fourth; and the fourth 6*d.* more than the youngest. They altogether earned 9*s.* 11*d.* a week. How much could each earn a week?

12. The Great Western Railway passes through Oxford, Leamington, and Birmingham, to Shrewsbury. From London to Shrewsbury is 171 miles. The distance from London to Oxford is 21 miles more, between Oxford and Leamington is 1 mile more, and between Leamington and Birmingham 19 miles less, than the distance between Birmingham and Shrewsbury. What are the distances between the successive stations?

13. What number is that, the treble of which is as much above 56, as its half is below it?

14. Two workmen received the same sum for their labour; but if one had received 11*s.* more, and the other 8*s.* less, then one would have had just twice as much as the other. What did each receive?

15. Two persons entered into a speculation by which one gained 45*l.* more than the other. The whole gain was 256*l.* less than three times the smaller gain. What were the respective gains?

16. The perimeter of a triangle is 69 yards. One side is 5 yards longer than the other, and is itself less than the base by 11 yards. Determine the lengths of the three sides.

17. A company settling their reckoning at a tavern pay 10*s.* each; had there been 6 less they would have paid 3*s.* a head more. How many were there?

18. Divide 51 into two such parts, that one of them being

divided by 9, and the other by 5, the quotients may together be equal to 7.

19. A pays 20*l.* more rent than B; but three times B's rent is 4*l.* more than twice A's. What rent does each pay?

20. Having bought a certain quantity of brandy at 21*s.* a gallon, and a quantity of rum, exceeding that of the brandy by 7 gallons, at 16*s.* a gallon, I find that I paid 3*s.* more for the brandy than for the rum. How many gallons were there of each?

21. Two persons, A and B, have each an annual income of 500*l.* A spends every year 50*l.* more than B, and in two years and a half the amount of their savings is equal to one year's income of either. What does each spend annually?

22. A draper sold two pieces of cloth, by one of which he gained 6*l.* more than by the other; and his whole gain was 7*l.* less than treble the smaller gain. How much did he gain by each piece?

23. In a naval engagement the number of ships taken was 9 more, and the number burnt 3 fewer, than the number sunk; 18 escaped, and the fleet consisted of 7 times the number sunk. Of how many ships did the fleet consist?

24. A cistern is filled in 19 minutes by 3 pipes, one of which conveys 12 gallons more, and the other 7 gallons less, than the third, per minute. The cistern holds 836 gallons. How much flows through each pipe in a minute?

25. A and B began to play; A with exactly three-eighths of the sum which B had. After winning 10*l.*, he found that they had each the same sum. What had each at first?

26. Two coaches at the distance of 200 miles set out to meet each other; one travels 9 miles an hour, the other only 7. What part of the distance does each travel?

27. Two persons, A and B, travelling, each with 100*l.*, meet with robbers, who take from A twice as much as from B, and 4*l.* over, and leave A 3*l.* more than the third of what B has left. How much is taken from each?

28. A person distributes 60*s.* amongst 50 people, giving to some 11*d.* each, and to the rest 1*s.* 4*d.* How many received the larger sum?

29. A regiment of militia, containing 660 men, is to be raised from 4 towns, A, B, C, D. The contingent of A is to be 3 times that of B, and that of C 4 times that of D, whilst this last furnishes one-sixth as many men as A and B together. Required the numbers raised by each.

30. Three persons, A, B, and C, spent equal sums at a tavern. C having no money, the reckoning was paid by A and B. When C came to reimburse them, he paid 5 times as much to A as to B; and observed, that if B had paid 6*s.* more of his reckoning, their demands would have been equal. Required the sum each spent, and the respective parts of C's reckoning that A and B paid.

31. A certain sum is divided amongst 3 persons; A receives 1,200*l.* less than the half, B 700*l.* more than the quarter, and C 1,500*l.* more than the fifth part of the whole. What is the sum divided, and what does each receive?

32. A besieged garrison had such a quantity of bread as would, if distributed to each at 11 ounces a day, last 3 weeks; but having lost 1,500 men in a sally, the governor was enabled to increase the allowance to 12 ounces per day for 4 weeks. Required the number of men at first in the garrison.

33. A composition of copper and tin, containing 100 cubic inches, weighed 505 ounces. How many ounces of each metal did it contain, supposing a cubic inch of copper to weigh $5\frac{1}{4}$ ounces, and a cubic inch of tin to weigh $4\frac{1}{4}$ ounces?

34. There are two towns, A and B, which are 103 miles distant from each other. A coach sets out from A at 7 o'clock in the morning, and travels at the rate of 7 miles an hour in the direct road towards B. At 11 o'clock a coach sets out from B to go to A, and goes at the rate of 8 miles an hour. Where will they meet?

35. At the review of an army the troops were drawn up in a solid mass, 50 deep, when there were just one-fourth as

many men in front as there were spectators. Had the depth, however, been increased by 5, and the spectators drawn up in the mass with the army, the number of men in the front would have been 120 fewer than before. Of how many men did the army consist?

SIMPLE EQUATIONS OF TWO UNKNOWN QUANTITIES.

EXERCISE XXVII.

1. EXPLAIN the usual method of solving simple equations of two unknown quantities.

2. Find the values of x and y in the following equations:

(1.) $\left.\begin{aligned} 2x - y &= 3 \\ 3x + y &= 7 \end{aligned}\right\}.$

(2.) $\left.\begin{aligned} 5x + 3y &= 19 \\ 5x - 2y &= 4 \end{aligned}\right\}.$

(3.) $\left.\begin{aligned} x + 3y &= 17 \\ 13x + 2y &= 110 \end{aligned}\right\}.$

(4.) $\left.\begin{aligned} 3x + 2y &= 23 \\ 5x - 3y &= 13 \end{aligned}\right\}.$

(5.) $\left.\begin{aligned} 7x + 9y &= 61 \\ 10y - 3x &= 57 \end{aligned}\right\}.$

(6.) $\left.\begin{aligned} 8x - 7y &= 9 \\ 9x - 2y &= 16 \end{aligned}\right\}.$

(7.) $\left.\begin{aligned} 6x - y &= 53 \\ 14y - 9x &= 8 \end{aligned}\right\}.$

(8.) $\left.\begin{aligned} 5x - 2y &= 7 \\ 9x + y &= 31 \end{aligned}\right\}.$

(9.) $\left.\begin{aligned} 3x + y &= 14 \\ 2x + 5y &= 18 \end{aligned}\right\}.$

(10.) $\left.\begin{aligned} 8x - 3y &= 11 \\ 5x - 11y &= 16 \end{aligned}\right\}.$

(11.) $\left.\begin{aligned} 14x + 5y &= 91 \\ 2y - x &= 10 \end{aligned}\right\}.$

(12.) $\left.\begin{aligned} 3x - 7y &= 17 \\ 11x + 5y &= 93 \end{aligned}\right\}.$

(13.) $\left.\begin{aligned} 2x &= 4 + 7y \\ 3x - 12y &= 3 \end{aligned}\right\}.$

(14.) $\left.\begin{aligned} 4x - 9y + 5 &= 0 \\ 7x - 5 &= 13y \end{aligned}\right\}.$

(15.) $\left.\begin{aligned} y(5+x) &= x(8+y) \\ 3x + 4 &= 4y - 13 \end{aligned}\right\}.$

(16.) $\left.\begin{aligned} 3x - 7y &= 14 \\ 11x &= 174 - 5y \end{aligned}\right\}.$

(17.) $\left.\begin{aligned} 45x + 8y &= 350 \\ 21x - 13y &= 132 \end{aligned}\right\}.$

(18.) $\left.\begin{aligned} 19x + 23y &= 221 \\ 18x + 24y &= 216 \end{aligned}\right\}.$

(19.) $\left.\begin{aligned} 28x - 9y &= 3 \\ 21x + 2y &= 81 \end{aligned}\right\}.$

(20). $\left.\begin{aligned} 35x - 18y &= 86 \\ 12x - 5y &= 16 \end{aligned}\right\}.$

(21.) $\left.\begin{aligned} 4x - 9y &= 51 \\ 8x + 13y &= 9 \end{aligned}\right\}.$

(22.) $\left.\begin{aligned} 7y - 13x &= 101 \\ 9x + 4y + 8 &= 0 \end{aligned}\right\}.$

(23.) $\left.\begin{aligned} 5x + 8y &= 17 \\ 4x - 3y &= 2 \end{aligned}\right\}.$

(24.) $\left.\begin{aligned} 9x + 8y &= 5 \\ 12(x - y) &= 1 \end{aligned}\right\}.$

(25.) $\left.\begin{aligned} 5x - 14y &= 4 \\ 15x + 21y &= 3 \end{aligned}\right\}.$

(26.) $\left.\begin{aligned} 13x - 2y &= 9 \\ 7x - 9y &= 8 \end{aligned}\right\}.$

(27.) $\left.\begin{aligned} \frac{x}{3} + \frac{y}{4} &= 5 \\ \frac{x}{4} - \frac{y}{2} &= 1 \end{aligned}\right\}.$

(28.) $\left.\begin{aligned} \frac{x}{15} + \frac{y}{10} &= 8 \\ \frac{x}{27} - \frac{y}{20} &= 1 \end{aligned}\right\}.$

(29.) $\left.\begin{aligned} \frac{x}{9} + \frac{y}{8} &= 86 \\ \frac{x}{8} + \frac{y}{9} &= 84 \end{aligned}\right\}.$

(30.) $\left.\begin{aligned} x - \frac{1}{7}(y - 3) &= 6 \\ 4y - \frac{1}{3}(x + y) &= 7 \end{aligned}\right\}.$

(31.) $\left.\begin{aligned} \frac{2x + 3y}{6} - 8 &= 0 \\ \frac{14y - 3x}{4} - y &= 11 \end{aligned}\right\}.$

(32.) $\left.\begin{aligned} 2(x - 3) - \frac{y - 2}{5} &= 4 \\ 3(y + 1) + \frac{x - 5}{3} &= 9 \end{aligned}\right\}.$

(33.) $\left.\begin{aligned} \frac{9 - 4x - 5y}{40} &= y - x \\ \frac{1 - 2x + y}{3} - 2y + \frac{3}{2} &= 0 \end{aligned}\right\}.$

(34.) $$\left.\begin{array}{l}\frac{1}{7}(2x-y)+3x=2y-7\\ \frac{1}{5}(y+5)+\frac{1}{6}(y-x+1)=2x-6\end{array}\right\}.$$

(35.) $$\left.\begin{array}{l}\frac{1}{7}(5x+4y)=5-\frac{4x-2y}{4}\\ \frac{5x-3y}{14}=\frac{1}{5}(4x+2y)-3\end{array}\right\}.$$

(36.) $$\left.\begin{array}{l}\frac{1}{11}(4x+2y)=6-\frac{1}{4}(5y-3x)\\ \frac{1}{3}(8y-10)=\frac{1}{6}(5x+3y)+5\end{array}\right\}.$$

(37.) $$\left.\begin{array}{l}\frac{6+x}{5}-\frac{2x-y-1}{4}=3y-8\\ \frac{5y-12}{2}+\frac{4x-7}{6}=23-5x\end{array}\right\}.$$

(38.) $$\left.\begin{array}{l}x-\frac{3y+4x}{7}=6\frac{6}{7}-\frac{9y+42}{14}\\ y-2-\frac{5x-4y-1}{2}=x-\frac{11y-x+3}{4}\end{array}\right\}.$$

(39.) $$\left.\begin{array}{l}ax+by=c\\ a'x+b'y=c'\end{array}\right\}.$$

(40.) $$\left.\begin{array}{r}x+y=p\\ mx-ny=0\end{array}\right\}.$$

(41.) $$\left.\begin{array}{l}\frac{x}{a}+\frac{y}{b}=m\\ \frac{x}{b}+\frac{y}{a}=n\end{array}\right\}.$$

(42.) $$\left.\begin{array}{l}\frac{a}{x}+\frac{b}{y}=p\\ \frac{b}{x}+\frac{a}{y}=q\end{array}\right\}.$$

3. Can the values of x and y be found from the equations

$$\left.\begin{array}{r}3x+15y=21\\ 2x+10y=14\end{array}\right\};$$

and if not, why not?

PROBLEMS PRODUCING SIMPLE EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES.

EXERCISE XXVIII.

1. THREE times the difference of two numbers is equal to their sum, and the smaller number increased by one is equal to two-thirds of the larger diminished by one. Find the numbers.

2. If two numbers are divided by 3, and 5, respectively, and the results added, their sum is equal to 27; but if the divisors are interchanged, the sum is 29. What are the two numbers?

3. A draper bought two pieces of cloth for 13*l.*, one being 7*s.*, and the other 8*s.* per yard. He sold them each at an advanced price of 2*s.*, and gained by the whole 3*l.* 10*s.* What were the lengths of the pieces?

4. What fraction is that, from the numerator of which if 3 be taken, its value is $\frac{1}{4}$; but if 2 be taken from the denominator its value is $\frac{1}{2}$?

5. A bill of 19*l.* 4*s.* was paid with half-sovereigns and florins, and the number of half-sovereigns exceeded the double of the number of florins by 1. How many were there of each?

6. Divide the numbers 35 and 40 into two such parts, that the sum of one out of each pair may be 47, and the difference of the others 2.

7. Two men, A and B, received 5*l.* for their wages, A having been employed 13, and B 16 days; and A received for working 7 days 4*s.* more than B did for 8 days. What were their daily wages?

8. Three inches have to be cut from the length, and two inches from the breadth, of a plate of glass, to make it fit a

frame, and it is found that 152 square inches of glass will be wasted; but if the breadth of the frame were one inch more, and the length one inch less, the waste would be 145 inches. Find the length and breadth of the frame.

9. A boy spends half-a-crown in apples and pears, buying his apples at 3, and his pears at 4 a-penny; afterwards he sells at the same rate one half of his apples and a quarter of his pears for one shilling and a halfpenny. How many did he buy of each?

10. There is a number consisting of two digits, which, when divided by the sum of the digits, gives the quotient 3; but if the digits be inverted, and the resulting number increased by 9, the quotient is also 9. Find the number.

11. Two persons, *A* and *B*, played cards, each with a different sum. After a certain number of games, *A* had won as much as he had at first, and found that if he had 3*s.* more, he would have had just three times as much as *B*. But *B* afterwards won 13*s.* back, and then had twice as much as *A*. What had each at first?

12. A farmer has for sale wheat of three different qualities, and has 11 quarters more of the worst, and 5 quarters less of the best than of the third quality. The price of the last, per quarter, is 6*s.* more than that of the worst, and 4*s.* less than that of the best, and the whole of the middle quality is sold for 12*l.* more, and 23*l.* 14*s.* less, than the best and worst qualities respectively. Find the number of quarters, and the price per quarter of each quality.

13. There is a number consisting of two digits, which, if divided by the sum of the digits, gives the quotient 6; but if the digits be inverted, and the resulting number divided by the sum of the digits diminished by 4, the quotient is 9. What is the number?

14. *A* and *B* can do a piece of work together in 10 days, which *B* working for 9 days, and *C* for 32, would together complete; in 8 days they would finish it, working all three together. In what time could they separately do it?

15. A person owes a certain sum to two creditors. At one time he pays them 67*l.*, giving to one half the sum due, and to the other two-fifths of his debt. At a second time he pays 23*l.*, giving the first two-sevenths, and the second one-third of the balances respectively due to them. What were the debts?

16. There is a cistern, into which the water is admitted by three cocks, two of which are of exactly the same dimensions. When they are all open, five-eighths of the cistern is filled in 9 hours; and if one of the equal cocks be stopped, seven-fifteenths of the cistern is filled in 9 hours and 36 minutes. In how many hours would each cock fill the cistern?

17. Find a number of three digits, the last two alike, and less than the first by 1, such that the number formed by writing the *single* digit in the middle may exceed that resulting from inverting the digits of the original number by half as much again as the smaller digit.

18. A farmer mixes barley at 4*s.* 8*d.* a bushel with rye at 6*s.* a bushel, and wheat at 8*s.* a bushel, so that the whole is 100 bushels, and worth 6*s.* 8*d.* a bushel. Had he put twice as much rye, and 10 bushels more of wheat, the whole would have been worth exactly the same per bushel. How much of each kind was there?

INVOLUTION AND EVOLUTION.

Exercise XXIX.

1. Explain what is meant by "*Involution.*" Deduce the rule for squaring any *simple* algebraical quantity. How must a fraction be squared?

2. What is a "*binomial quantity?*" Write down the

squares of $a + b$, and of $a - b$. Express your results in words, and deduce a rule for squaring any binomial.

3. What is meant by a "*complete square?*" Can a quantity consisting of two terms be a complete square? Give a reason for your answer.

4. What is the test for discovering whether or not any proposed *trinomial* is a complete square? Is $x^2 + 4x + 1$ a complete square?

5. If y added to $x^2 + px$, or $x^2 - px$, make the resulting trinomial a complete square, show that $y = \frac{p^2}{4}$.

State in words the rule for completing a square of which the first two terms are given.

6. Explain the meaning of "*Evolution.*" What is the rule for extracting the square root of a *simple* algebraical quantity? Can the square root of a *binomial* be extracted?

7. When a trinomial has been ascertained to be a complete square, what is the rule for writing down its square root?

8. Explain how the *double sign* arises in extracting square roots.

9. How is the square root of a fraction found?

10. Square each of the following:

(1.) $2ab$.
(2.) $3abx$.
(3.) $4a^2y$.
(4.) $-5xy^2z$.
(5.) $7m^2np^3$.
(6.) $\frac{x}{a}$.
(7.) $\frac{c}{ab}$.
(8.) $-\frac{ac}{b}$.
(9.) $\frac{3ab^2}{4c^3}$.
(10.) $\frac{5}{6xy^2z^3}$.
(11.) $1 + x$.
(12.) $xy - 1$.
(13.) $a + 4$.

(14.) $3 - b$.

(15.) $2m + n$.

(16.) $3m - 2n$.

(17.) $x - \frac{a}{2}$.

(18.) $2x + \frac{y}{4}$.

(19.) $x^2 + p$.

(20.) $\frac{1}{2}y^2 + xy$.

11. Extract the square root of each of the following quantities:

(1.) $9a^2x^2$.

(2.) $25a^2x^2y^2$.

(3.) $16x^2y^4$.

(4.) $36a^6b^2$.

(5.) $\frac{a^2x^2}{y^2}$.

(6.) $\frac{49x^4y^6}{64z^2}$.

(7.) $\frac{9}{16}\left(\frac{x}{y}\right)^2$.

(8.) $\left(\frac{4a^2}{5b^2}\right)^2$.

(9.) $a^2 - 2ax + x^2$.

(10.) $x^2 + xy + \frac{1}{4}y^2$.

(11.) $1 - 2m + m^2$.

(12.) $\frac{a^2}{4} - a + 1$.

(13.) $\frac{x^2}{4} + \frac{1}{3}xy + \frac{y^2}{9}$.

(14.) $x^2 + 2 + \frac{1}{x^2}$.

(15.) $\frac{x^2}{y^2} + \frac{y^2}{x^2} - 2$.

(16.) $x^4 + 4x^2y^2 + 4y^4$.

12. Complete the squares in the following expressions, and write down the square root of the result in each case:

(1.) $x^2 + 6x$.

(2.) $x^2 - 18x$.

(3.) $x^2 - 5x$.

(4.) $x^2 + x$.

(5.) $x^2 - \frac{2}{3}x$.

(6.) $x^2 + \frac{3}{5}x$.

(7.) $x^2 - \frac{1}{2}x$.

(8.) $x^2 + 2mx$.

(9.) $x^2 - px$.

(10.) $x^2 - \frac{3}{4}nx$.

(11.) $a^2x^2 + 2abx$.

(12.) $a^2x^2 - abxy$.

QUADRATIC EQUATIONS.

EXERCISE XXX.

1. WHAT is a *Pure Quadratic Equation?* What an *Adfected Quadratic?*

2. How is a pure quadratic solved? Solve the equation $5x^2 - 8 = 3x^2 + 24$.

3. Explain fully the process by which an adfected quadratic is solved.

4. Write down the *general form* to which all adfected quadratics must be reduced, and then obtain the *general solution.*

5. Solve the following pure quadratics:

(1.) $6x^2 - 10 = 5x^2 + 15$.

(2.) $\frac{x^2}{3} + \frac{x^2}{6} = 5\frac{1}{2} - \frac{1}{9}x^2$.

(3.) $\frac{2x^2}{7} + 4 = x^2 - \frac{x^2 - 1}{6} - 23$.

(4.) $10\left(\frac{x^2+1}{2} - 1\right) - 3x^2\left(\frac{2}{x^2} - \frac{1}{3}\right) = 13$.

6. Solve the following adfected quadratics:

(1.) $x^2 - 2x = 3$.

(2.) $x^2 + 10x = -9$.

(3.) $x^2 - 3x + 2 = 0$.

(4.) $x^2 + 7x = 128 - x$.

(5.) $32x = 320 - x^2$.

(6.) $x^2 - \frac{x}{2} = 3$

(7.) $7x + 3x^2 = 48$.

(8.) $5 - 2x^2 = 9x$.

(9.) $2x^2 + 1 = 11(x + 2)$.

(10.) $\frac{3}{4}(x^2 - 3) = 15 + \frac{1}{2}(x - 2)$.

7. Solve the following equations:

(1.) $12x^2 - 32 = 9x^2 + 16.$

(2.) $3x^2 - 28 = x^2 - 20.$

(3.) $x^2 + 2x = 143.$

(4.) $4x^2 + 4 = 32 - 3x^2.$

(5.) $x^2 - 6x + 3 = 75.$

(6.) $x^2 + 19 = 3x^2 - \frac{6 - x^2}{3}.$

(7.) $x^2 - 10x + 17 = 1.$

(8.) $\frac{3x^2 + 5}{8} - \frac{21 + 4x^2}{3} = 39 - 6x^2$

(9.) $x^2 - 2x = 0.$

(10.) $x^2 - 3x - 10 = 0.$

(11.) $x^2 - 40 = x + 2.$

(12.) $x^2 + 9x + 14 = 0.$

(13.) $x^2 - 3x = 5x - 16.$

(14.) $7(2x^2 - 4) + 32 = 8(2x^2 - 2) + 3(3x^2 - 8).$

(15.) $(x - 3)^2 = 34 - 6x.$

(16.) $x^2 - x - 40 = 170.$

(17.) $\frac{x^2}{12} + \frac{5}{3} = x.$

(18.) $5x^2 + \frac{1}{4} = 9x - 2.$

(19.) $\frac{2x^2}{3} + 3\frac{1}{2} = \frac{x}{2} + 8.$

(20.) $x + 5 + \frac{7x - 2}{x} = 13.$

(21.) $4x - \frac{35 - x}{x} = 24.$

(22.) $\frac{x + 5}{2} + \frac{12 - 2x}{2x - 1} = 5\frac{1}{5}.$

(23.) $\frac{2x - 8}{7 - x} - \frac{x + 4}{x - 1} = 2.$

(24.) $14 + 4x - \frac{x + 7}{x - 7} = 3x + \frac{9 + 4x}{3}.$

(25.) $\frac{x + 2}{x - 1} - \frac{4 - x}{2x} = \frac{7}{2}.$

(26.) $\frac{4x - 5}{x} - \frac{3x - 7}{3x + 7} = \frac{9x + 23}{13x}.$

(27.) $\frac{8}{9 + 5x} + \frac{8x - 17}{2 + 4x} = \frac{4x + 3}{2x + 12}.$

(28.) $x^2 - (a + b)x + ab = 0.$

(29.) $x - \frac{x^3 - 8}{x^2 + 5} = 2.$

EXERCISE XXXI.

OBSERVATION.—The simultaneous Equations of the following Exercise may all be solved by deriving, from one of the Equations, the value of one of the unknown quantities in terms of the other, and substituting this value in the second of the proposed Equations. The student will occasionally, however, derive great assistance from observing the method employed in the solution of the subjoined examples.

(1.) Given $\left.\begin{aligned} x + y &= 4 \\ xy &= 3 \end{aligned}\right\}$: to find x, and y.

Squaring the first equation, we obtain

$$x^2 + 2xy + y^2 = 16,$$

multiplying the second equation by 4, $\quad 4xy = 12,$

whence by subtraction, $\quad x^2 - 2xy + y^2 = 4,$

$$\text{and } \therefore\ x - y = \pm 2.$$

Adding this to the first of the proposed equations, we get

$$2x = 6, \text{ or } 2,$$

$$\text{and } \therefore\ x = 3, \text{ or } 1;$$

and by subtraction, $2y = 2$, or 6,

$$\text{and } \therefore\ y = 1, \text{ or } 3.$$

(2.) Given $\left.\begin{aligned} x - y &= 7 \\ xy &= 18 \end{aligned}\right\}$: to find x, and y.

The student will readily see that the solution of this equation will be obtained by proceeding in a manner similar to that employed in (1), the only difference being, that, having given $x - y$, we must *add* to its square $4xy$ to obtain $x + y$.

Solve the following simultaneous equations:

(1.) $\left.\begin{aligned} x^2 + y^2 &= 13 \\ x^2 - y^2 &= 5 \end{aligned}\right\}.$

(2.) $\left.\begin{aligned} x + 3y &= 15 \\ y^2 - x &= 13 \end{aligned}\right\}.$

(3.) $\left.\begin{aligned} 2x + 3y &= 20 \\ 2x^2 - 15y^2 &= 38 \end{aligned}\right\}.$

(4.) $\left.\begin{aligned} x - y &= 5 \\ xy &= 36 \end{aligned}\right\}.$

(5.) $\left.\begin{aligned} x + y &= 10 \\ xy &= 9 \end{aligned}\right\}.$

(6.) $\left.\begin{aligned} x + y &= 7 \\ x^2 + y^2 &= 25 \end{aligned}\right\}.$

(7.) $\left.\begin{aligned} x - y &= 2 \\ x^2 - y^2 &= 16 \end{aligned}\right\}.$

(8.) $\left.\begin{aligned} x - y &= 1 \\ x^2 + y^2 &= 5 \end{aligned}\right\}.$

(9.) $\left.\begin{aligned} xy &= 6 \\ x^2 + y^2 &= 13 \end{aligned}\right\}.$

(10.) $\left.\begin{aligned} 2x + y &= 7 \\ 4x^2 + y^2 &= 25 \end{aligned}\right\}.$

(11.) $\left.\begin{aligned} \frac{2}{x} + \frac{3}{y} &= 8 \\ 7xy &= 6 \end{aligned}\right\}.$

(12.) $\left.\begin{aligned} x + 4y &= 23 \\ y^2 + 4x &= 2y + 27 \end{aligned}\right\}.$

(13.) $\left.\begin{aligned} \frac{3x}{7y} &= \frac{6}{7} \\ x + 2y &= xy \end{aligned}\right\}.$

(14.) $\left.\begin{aligned} x^2 - xy - y^2 &= \frac{1}{15}xy \\ x - y &= 2 \end{aligned}\right\}.$

(15.) $\left.\begin{aligned} 2x^2 - 3xy + y^2 &= 2 \\ 3x^2 - 5xy + 2y^2 &= 6 \end{aligned}\right\}.$

(16.) $\left.\begin{aligned} \frac{2x + 7y}{4x} &= 2y - \frac{51 + 2x}{10} \\ 4x + 3y &= 16(y - 2) \end{aligned}\right\}.$

(17.) $\left.\begin{aligned} \frac{y}{x + 5} &= \frac{y}{x} - 21 \\ \frac{y}{x - 3} &= \frac{y}{x} + 21 \end{aligned}\right\}.$

(18.) $\left.\begin{aligned} x + y &= a \\ x^2 + y^2 &= b^2 \end{aligned}\right\}.$

(19.) $\left.\begin{aligned} xy &= a^2 \\ x + y &= b \end{aligned}\right\}.$

(20.) $\left.\begin{aligned} xy &= a^2 \\ x - y &= b \end{aligned}\right\}.$

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

Exercise XXXII.

1. There are two numbers, one of which is two-thirds of the other, and their product is 96. Find the numbers.

2. Required the two numbers, whose difference is equal to 6, and whose product is 40.

3. In a court there are two square grass plots: a side of

one of which is 7 yards longer than that of the other, and the area of the smaller is $\frac{4}{9}$ths of that of the larger. What are the lengths of the sides?

4. The sum of two numbers is 12, and the quotient of the larger by the smaller is half the latter. Find the numbers.

5. Divide the number 24 into two such parts, that one may be half the square of the other.

6. The difference between the length of the sides of a rectangular field is 20 yards, and the distance between two opposite corners is 100 yards. Find the lengths of the sides of the field, and its area.

7. A pedestrian having to walk 36 miles, finds that if he increases his speed 1 mile per hour, he will perform his journey in 3 hours less than if he walked at his usual rate. What is that rate?

8. A charitable person is about to distribute 6*l.* among some poor persons, when two others come in, and thus each person's share is diminished by 2*s.* Find the number of persons relieved.

9. A gardener and his lad dug each a square piece of ground, of which the side was as many feet long as the worker was years old. The difference of their ages was 12 years, and the number of square feet dug by both was 1,040. Required their ages.

10. A person bought a quantity of cloth for 33*l.* 15*s.*, which he sold at 2*l.* 8*s.* per piece, and gained by the bargain as much as one piece cost him. Find the number of pieces.

11. Two detachments of foot, being ordered to a station at the distance of 39 miles from their present quarters, began their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour more than the other, arrived there an hour sooner. Find their rates of marching.

12. There is a number consisting of two digits, the left hand digit being three times the other; and if 12 be subtracted

from the number, the remainder will be the square of the left hand digit. Find the number.

13. A merchant sold a quantity of brandy for 24*l.* and gained as much per cent. as it cost him. What was the cost of the brandy?

14. There are two numbers of which the product is 120; and if 2 be added to the less, and 3 subtracted from the greater, the product of the sum and difference will be also 120. What are the numbers?

15. What number is that, which, being divided by the product of its two digits, the quotient is 2, and if 27 be added to the number, the digits will be inverted?

16. A grocer sold 80 pounds of mace and 100 pounds of cloves for 65*l.*, and finds that he has sold 60 pounds more of cloves for 20*l.* than of mace for 10*l.* What was the price of a pound of each?

17. A person bought a certain number of oxen for 285*l.*, and, after losing 4, sold the remainder for 7*l.* a head more than they cost him, thus gaining 45*l.* by the bargain. How many did he purchase?

18. A man has to travel a certain distance; when he has travelled 40 miles he increases his speed 2 miles per hour; if he had travelled with his increased speed during the whole of his journey he would have arrived 40 minutes earlier; but if he had continued at his original speed he would have arrived 20 minutes later. How far had he to travel, and at what rate did he start?

19. Find three numbers whose sum is 19, such that the difference of the first and second is double that of the second and third, and the sum of whose squares is 139.

20. *A* and *B* were going to market, the first with cucumbers, and the second with three times as many eggs; and they find that if *B* gave all his eggs for the cucumbers, *A* would lose 10*d.* according to the rate at which they were then selling. *A* therefore reserves two-fifths of his cucumbers; by which *B* would lose 6*d.* according to the same rate. But *B*, selling

the cucumbers at 6*d.* a piece, gains upon the whole the price of 6 eggs. Find the number of eggs, and cucumbers, and their price.

RATIO AND PROPORTION.

Exercise XXXIII.

1. Define a "*Ratio*," and explain how it is measured. Write algebraically *the ratio of* a *to* b.

2. Explain what is meant by a "*Proportion.*" Illustrate by an example the mode in which a proportion is usually expressed algebraically, and write your proportion in *words* as it should be read.

3. How is a proportion converted into an equation?

4. How can we determine which is the greater of two proposed ratios?

5. Show that a ratio is not altered in value by having both its terms multiplied, or divided by the same quantity.

6. In a proportion show that the product of the extreme terms is equal to that of the means, and hence deduce the Single Rule of Three.

7. If $a : b :: c : d$, prove the following results:

(1.) $b : a :: d : c$.
(2.) $a : c :: b : d$.
(3.) $a + b : b :: c + d : d$.
(4.) $a + b : a - b :: c + d : c - d$.

8. If $a : b :: c : d$, and $c : d :: e : f$; show that $a : b :: e : f$.

If $a : b :: c : d$, and $b : e :: d : f$; prove that $a : e :: c : f$.

9. Give Euclid's definition of proportion, and show that quantities which are proportional according to the algebraical definition are proportional also according to the geometrical definition.

10. Find the value of each of the following ratios:

(1.) $2x : 6x$
(2.) $3a : 21a^2$.
(3.) $21a^2 : 3a$.
(4.) $mx : nx$.
(5.) $3a : 7ax$.
(6.) $2max : 3nay$.
(7.) $5abc : 15ab^3$.
(8.) $6x^2yz : 9xyz^2$.
(9.) $ax + bx : 2xy$.
(10.) $y^2 - y : ay^2$.
(11.) $3x^2 - 2ax : 2x^2 - 3ax$.
(12.) $x(x-y) : x^2 - y^2$.
(13.) $x^2 - y^2 : a(x+y)$.
(14.) $x^3 + y^3 : x^2 - xy + y^2$.
(15.) $ax + bx + cx : ax - bx + cx$.
(16.) $\frac{1}{2}(x^2 + y^2) : \frac{1}{3}(x^4 - y^4)$.

Simplify the following ratios:

(1.) $3ax : 2a$.
(2.) $8a^2x : 14ax^2$.
(3.) $6abx : 4by$.
(4.) $\frac{1}{3}axy : \frac{1}{2}ay^2$.
(5.) $\frac{1}{4}ax^5y : \frac{5}{6}a^2x^3$.
(6.) $a^2 - b^2 : a - b$.
(7.) $n^3 + 1 : n^2 + n$.
(8.) $x^2 - y^2 : (x+y)^2$.

11. Determine which is the greater of the ratios, $7 : 8$, and $10 : 11$; $19 : 25$, and $56 : 74$.

12. What is the difference of the values of the ratios, $13 : 14$, and $23 : 24$?

13. If a is greater than b, show that the ratio $a + b : a - b$, is greater than $a^2 + b^2 : a^2 - b^2$.

14. If $a : b :: c : d$, prove that

$$ma + nb : ma - nb :: pc + qd : pc - qd.$$

15. If $a : b = b : c$, show that $a : c = a^2 : b^2$.

16. Find the value of x, when $1 - x : 1 + x :: 1 : 3$.

17. Convert the proportions

$$a : b + x :: b - x : a,$$
$$3 : x :: x - 2 : 8,$$

into equations.

18. If $a + x : a - x :: 7 : 3$, find the value of $x : a$.

19. Find the 1st, 2d, 3d, and 4th terms of the 4 proportions in which the other terms taken in order are

$$3ax, \quad 4by, \quad \frac{5}{6}xy.$$

20. Two numbers are in the ratio of $3 : 4$, and their difference : their product $:: 1 : 24$. Find the numbers.

21. If the terms of the ratio $7 : 9$ are both increased by a certain quantity, the resulting numbers are in the ratio $5 : 6$. Determine this quantity.

VARIATION.

Exercise XXXIV.

1. What is meant by a *variable* quantity? When two quantities are mutually dependent on each other, do they necessarily vary as each other? When does one quantity *vary directly* as another?

2. What is meant by the *reciprocal* of a quantity? When does one quantity *vary inversely* as another?

3. When does one quantity vary as two others jointly?

4. Show how, when two sets of corresponding values of the variables are known, every variation may be converted into an equation.

5. If $y \propto x$, and when $x = 4$, $y = 12$; find the equation between x and y.

6. If $y \propto x$, and when $x = 2$, $y = 5$; find the value of y when $x = 8$.

7. If $y^2 \propto x$, and when $x = a$, $y = \pm 2a$; find the equation between x and y.

8. If $y \propto \frac{1}{x}$, and when $x = \frac{1}{3}$, $y = 6$; find the equation between x and y.

9. If $y \propto \frac{3}{x^2}$, and when $x = 2$, $y = 12$; find the value of y when $x = \frac{1}{4}$.

10. If $z \propto x + y$, and $y \propto x^2$, and $z = 3$ when $x = 1$, $y = 2$; find the relation between z and x.

11. If $x \propto u$, and $u \propto y$; then $u \propto \sqrt{xy}$.

12. If $xy \propto x^2 + y^2$, and when $x = 3$, $y = 4$; find the equation between x and y.

13. If $y \propto x$, show that $ax + by \propto mx + ny$, the coefficients a, b, m, n being any constants whatever.

14. If a gold coin were made of the same thickness as a half-sovereign, and of the value of a half-crown, what would be the ratio of the diameters of such a coin and a half-sovereign? (N.B. The area of a circle varies as the square of its radius.)

ARITHMETICAL PROGRESSION.

Exercise XXXV.

1. When are quantities said to be in *Arithmetical Progression?* What is meant by the "*common difference?*" Give examples both of *increasing* and *decreasing* arithmetical series.

2. Which of the following series are in arithmetical progression?

(1.)	4,	7,	9,	12,	&c.
(2.)	3,	7,	11,	15,	&c.
(3.)	17,	15,	13,	11,	&c.
(4.)	12,	9,	6,	4,	&c.

3. Represent an arithmetical progression generally by means of algebraical symbols, and deduce the expression for the nth term of the series.

4. From what *data* can any proposed term of an arithmetical progression be determined without knowing all the preceding terms?

Find the 30th term of the series, 2, 5, 8, &c.

5. State and prove the rule for finding the sum of any given number of consecutive terms of an arithmetical progression.

Find the sum of 12 terms of the series, 3, 5, 7, &c.

6. What is meant by "*the Arithmetic Mean*" between two quantities? Show that it is equal to half the sum of the two quantities.

7. What is meant by inserting any given number of *arithmetic means* between two quantities? Show how to do it.

Insert three arithmetic means between 7 and 27.

8. Find the 13th term of the series 3, 7, 11, &c.

9. Find the 15th term of the series 5, 1, − 3, &c.

10. Find the 20th term of the series − 4, − 1, 2, &c.

11. Find the 10th, 14th, and 30th terms of the series

$$\frac{2}{3}, \frac{7}{12}, \frac{1}{2}, \text{ \&c.}$$

12. If the 1st term of an arithmetic series be 5, and the 4th term 17; find the 13th, 16th, and 23d terms.

13. The 3d and 7th terms of an arithmetic series are 13, and 21, respectively; find the 1st term, and common difference.

14. Write down the series whose 4th and 9th terms are respectively 11 and $28\frac{1}{2}$.

15. Find the value of

(1.) $1 + 3 + 5 +$ &c. to 18 terms.

(2.) $2 + 4 + 6 +$ &c. to 19 terms.

(3.) $3 + 2 + 1 +$ &c. to 5 terms, to 7 terms, and to 10 terms.

(4.) $-5 - 1 + 3 +$ &c. to 22 terms.

(5.) $-3 - 6 - 9 -$ &c. to 8 terms.

(6.) $2 + 2\frac{1}{4} + 2\frac{1}{2} +$ &c. to 15 terms.

(7.) $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} +$ &c. to 12 terms.

(8.) $6 + \frac{11}{2} + 5 +$ &c. to 23 terms.

(9.) $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} -$ &c. to 17 terms.

(10.) $1 + 3 + 5 +$ &c. to n terms.

(11.) $\frac{1}{3} + \frac{5}{6} + \frac{4}{3} +$ &c. to n terms.

(12.) $\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} +$ &c. to 5 terms, and also to n terms.

(13.) $\frac{n-1}{n} + \frac{n-3}{n} + \frac{n-5}{n} +$ &c. to n terms. How many terms of the series $\frac{7}{8}, \frac{5}{8}, \frac{3}{8}$, &c. amount to 0?

16. If s be the sum of n terms of an arithmetic series whose first term is a, show that the common difference $= \frac{2(s - na)}{n(n-1)}$.

17. The sum of 30 terms of an arithmetic series is 2,235, and the 1st term is 2; find the common difference.

18. The 5th term of an arithmetic series is 25, and the sum of 13 terms is 481; find the 1st term, and the common difference.

19. The sum of n terms of an arithmetic series, whose 1st term is 3, and common difference 7, is 279; find n.

20. How many terms of the series 54, 51, 48, &c. amount to 513? Explain the double answer.

21. The 1st term of an arithmetic series is 19, the last term $\frac{5}{2}$, and the number of terms 12; find the common difference.

22. Find the arithmetic mean between 6 and 20; between $4\frac{1}{2}$ and 7; and between 3 and -5.

23. Insert two arithmetic means between 9 and 21; three between 102 and 138.

24. Insert seven arithmetic means between 1 and -1.

25. Find the sum of n terms of the series

$$2a + 3x, \quad a + 4x, \quad 5x, \text{ \&c.}$$

26. There are three numbers in arithmetical progression whose sum is 21; and the sum of the first and second is to the sum of the second and third as 5 to 9. Required the numbers.

27. A number consisting of three digits which are in arithmetical progression, being divided by the sum of its digits, gives a quotient 59; and if 396 be subtracted from it, the digits will be inverted. Required the number.

28. A person bought three horses, the prices given for them being in arithmetical progression, and sold the best and worst for a quarter as much again as he gave for the three. By the whole transaction he found that he got the horse he

kept for himself for nothing, and gained 30*l.* beside. What did he give for his own horse?

29. Two boys set out from Shrewsbury and Rugby to meet each other. Supposing the distance between these two places to be 75 miles, and that one boy walks 3 miles the first day, 5 the next, 7 the third, and so on; and the other 4 miles the first day, 6 the next, and so on; in how many days will they meet?

GEOMETRICAL PROGRESSION.

Exercise XXXVI.

1. When are quantities said to be in *Geometrical Progression?* What is "*the Common Ratio?*" Give instances both of *increasing* and *decreasing* geometric series.

2. Which of the following series are in geometrical progression?

(1.) $1, \quad 4, \quad 16, \quad 64,$ &c.
(2.) $3, \quad 6, \quad 10, \quad 15,$ &c.
(3.) $1, \quad -2, \quad -4, \quad 8,$ &c.
(4.) $\frac{1}{2}, \quad 1\frac{1}{2}, \quad 4\frac{1}{2}, \quad 13\frac{1}{2},$ &c.

3. What are the common ratios of the following series?

$$1, \quad -\frac{1}{2}, \quad \frac{1}{4}, \text{ \&c.}$$

$$3\tfrac{3}{8}, \quad 2\tfrac{1}{4}, \quad 1\tfrac{1}{2}, \text{ \&c.}$$

4. Represent a series of terms in geometrical progression by means of general algebraical symbols, and deduce the expression for the nth term of the series.

5. From what *data* can any proposed term of a geometric series be found independently of the rest?

Find the 8th term of the series $\frac{1}{2}$, 1, 2, &c.

6. State and prove the formula for finding the sum of any given number of consecutive terms of a geometric series.
Find the sum of 6 terms of the series 3, 6, 12, &c.

7. What is meant by *the Geometric Mean* between any two quantities? Show that it is equal to the square root of their product.

8. What is meant by inserting any given number of geometric means between two quantities? Show (1) how two, (2) how n geometric means may be inserted between a and b. Insert two geometric means between 1 and 27.

9. Find the common ratio and 5th term of the series

(1.) $1\frac{1}{4}$, $2\frac{1}{2}$, 5, &c. (2.) 27, -45, 75, &c.

10. Find the 6th term of the series 1, 3, 9, &c.

11. Find the 4th and 6th terms of the series of which the 1st and 2d terms are $3\frac{3}{8}$ and $2\frac{1}{4}$.

12. Find the 5th term in each of the following geometric series:

(1.) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c.

(2). $-2, \frac{3}{2}, -1\frac{1}{8}$, &c.

(3.) $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}$, &c.

(4.) $a, b, \frac{b^2}{a}$, &c.

(5.) $\frac{x}{y}, -\frac{y}{x}, \frac{y^3}{x^3}$, &c.

13. Write down the geometric series whose 4th and 5th terms are 125 and $312\frac{1}{2}$.

14. Find the value of

(1.) $1 + 3 + 9 +$ &c. to 6 terms.

(2.) $1 + \frac{1}{2} + \frac{1}{4} +$ &c. to 7 terms.

(3.) $9 - 6 + 4 -$ &c. to 5 terms.

(4.) $\frac{1}{3} + \frac{1}{2} + \frac{3}{4} +$ &c. to 7 terms.

(5.) $\frac{16}{27} + \frac{4}{9} + \frac{1}{3} +$ &c. to 5 terms.

15. The 2d term of a geometric series is $-\frac{1}{4}$, and the 5th is $\frac{27}{256}$; find the sum of the first 4 terms.

16. Find the sum of 6 terms of the series of which the first two terms are 4 and 2.

17. Find the geometric mean between 4 and 16; between 2 and 18; and between 8 and $4\frac{1}{2}$.

18. Find the geometric mean between $\frac{3}{4}$ and $1\frac{2}{25}$.

19. Insert two geometric means between 8 and 125.

20. Insert two geometric means between $3\frac{3}{8}$ and 1.

21. Insert three geometric means between 4 and 324.

22. Insert three geometric means between $\frac{1}{2}$ and $\frac{8}{81}$.

23. Insert four geometric means between 4 and $\frac{1}{8}$.

24. Which is greater, the arithmetic or the geometric mean between $\frac{1}{2}$ and $\frac{1}{8}$? and by how much greater?

25. The sum of three numbers in geometrical progression is 21, and the sum of the first and second is to the sum of the second and third as 1 to 2. Required the numbers.

26. There are three numbers in geometrical progression; the sum of the first and second is 6, and the sum of the first and third is 10. Find the numbers.

HARMONICAL PROGRESSION.

Exercise XXXVII.

1. When are quantities said to be in *Harmonical Progression?* Give an example of a harmonic series.

2. Which of the following series are in harmonical progression?

(1.) 2, 3, 6, &c.
(2.) 3, 4, 7, &c.
(3.) 5, 4, $3\frac{1}{3}$, &c.
(4.) $\frac{2}{5}$, $\frac{2}{3}$, $2\frac{1}{2}$, &c.

3. Represent a series of terms in harmonical progression by means of general algebraical symbols, and deduce the expression for the nth term of the series.

4. From what *data* can any proposed term of a harmonic series be found independently of the rest?
Find the 5th term of the series 5, 6, $7\frac{1}{2}$, &c.

5. Can a general expression be found for the *sum* of a series of terms in harmonical progression, as in the case of arithmetic and geometric series?

6. What is *the Harmonic Mean* between two quantities? Find the harmonic mean between a and b.

7. What is meant by inserting any given number of *harmonic means* between two quantities? Show how to do it.

8. Find the 6th term of the series $2, \frac{1}{2}, \frac{2}{7}$, &c.

9. Find the 10th term of the series $1\frac{1}{2}$, $2\frac{1}{7}$, $3\frac{3}{4}$, &c.

10. Find the 5th term of the series $\frac{3}{5}, 2, -\frac{3}{2}$, &c.

11. Find the harmonic means between 2 and 6; $\frac{1}{6}$ and $\frac{1}{5}$; $2\frac{2}{7}$ and 16.

12. Insert two harmonic means between 5 and $1\frac{1}{4}$.

13. Insert three harmonic means between $\frac{1}{2}$ and $\frac{1}{3}$; four between $\frac{1}{15}$ and $-\frac{1}{5}$.

14. Show that the arithmetical, geometrical, and harmonical means between a and b form a geometrical progression.

15. Show that three quantities, a, b, c, are in arithmetical, geometrical, or harmonical progression, according as

$$\frac{a-b}{b-c} = \frac{a}{a}, \text{ or } = \frac{a}{b}, \text{ or } = \frac{a}{c}.$$

16. The harmonic mean between two numbers is $\frac{3}{4}$ of the arithmetic mean; show that one of the numbers is equal to three times the other.

APPENDIX.

MISCELLANEOUS EXAMPLES.

I.

1. If $a = 1$, $b = 2$, $c = 3$, prove that
$$6abc = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc,$$
and also $= a^3 + b^3 + c^3$.

2. Subtract
$$2a^3 - 3a^2b + ab^2 + b^3 \text{ from } 5a^3 + a^2b - 6ab^2 + b^3.$$

3. Find the continued product of $x - a$, $x - b$, and $x - c$.

4. Reduce to lowest terms $\dfrac{4x^4(a+x)^2}{10(a^2x - x^3)^2}$.

5. Find the least common multiple of $x^2 - 1$, $(x-1)^2$, and $x^2 + 1$.

6. Solve the following equations:

(1.) $\dfrac{5x}{9} - \dfrac{2x-1}{3} = \dfrac{4}{15}$. (2.) $5x^2 + \dfrac{9}{4} = 9x$.

(3.) $\left.\begin{aligned} 2x - y &= 8 \\ 2y + x &= 9 \end{aligned}\right\}$.

7. The hour and minute hand of a watch are at right angles, and it is between 4 and 5 o'clock. Determine the precise time.

8. What quantity must be added to each of the terms of the ratio 3 : 5, that it may become equal to the ratio 5 : 6?

9. Given that $y \propto x$, and that when $x = 2$, $y = 7$, find the value of y when $x = 13$.

10. Sum the series $1 + \frac{1}{2} + \frac{1}{4} +$ &c. to 10 terms; and show that 7 times any one term is equal to 8 times the sum of the three succeeding ones.

II.

1. Find the value of $\frac{x+y}{x-y}$, and of $\frac{x}{y} - \sqrt{\frac{1+x}{1-y}}$, when $x = \frac{1}{4}$, $y = \frac{1}{5}$.

2. Add together $m^2 + 2mp - n^2$, $3m^2 + 2n^2$, and $q^2 - 4mp$.

3. Divide $x^4 - y^4$ by $x - y$; $x^3 - apx^2 + a^2px - a^3$ by $x - a$.

4. Find the least common multiple of $3a^2x$, $6x^4y$, and $14a^3y^2$.

5. Multiply $x + \frac{1}{x}$ by $x - \frac{1}{x}$; and reduce $\frac{ax + x^2}{3bx - cx}$ to lowest terms.

6. Simplify

$$\frac{x}{2} - \left\{x - \frac{3x-1}{4}\right\}; \text{ and } a - [b - \{c - (d+e)\}].$$

7. Solve the following equations:

$$(1.)\ \frac{3x-4}{7} - \frac{1}{2}\cdot(x - 17) = 6\tfrac{1}{2}.$$

$$(2.)\ 3x^2 - 4x = 15 - \frac{20x^2 - 3}{7}.$$

$$(3.)\ \left.\begin{aligned} \frac{x}{2} + \frac{7y}{3} &= 10 \\ \frac{2x}{3} - y &= 1 \end{aligned}\right\}.$$

8. Find the number which exceeds its fifth part by 24.

9. If $x \propto y$, and when $x = \frac{1}{2}$, y be equal to $5x - 1$; find the equation between x and y.

10. Sum the series $2 + 5 + 8 +$ &c. to 11 terms; and find the twentieth term.

III.

1. If $a = 2$, $b = 3$, $c = 4$, show that

$$4a^2b^2 - (c^2 - a^2 - b^2)^2 = 4b^2c^2 - (a^2 - b^2 - c^2)^2.$$

2. Add together $ax + by + cz$, $ax - 2by - cz$, $2by - 2ax + 3cz$, $5ax + 4by - 6cz$, and $ax + by + 9cz$.

3. Multiply $x^2 - px + q$ by $x + a$; and divide $x^4 - y^4$ by $x + y$.

4. Reduce $\frac{a^3 - x^3}{(a - x)^3}$ to its lowest terms; and simplify the expression $\frac{1}{x - 1} - \frac{7}{2x - 4} + \frac{7}{2x - 8}$.

5. Prove that $a - \{b - (c - d)\} = a - b + c - d$.

6. Solve the equations:

(1.) $$\frac{x - 3}{4} - \left(\frac{x - 5}{6} + \frac{x - 1}{9}\right) = 0.$$

(2.) $$\frac{2x + 11}{x} = 5 - \frac{x - 5}{3}.$$

(3.) $$\left.\begin{aligned} \frac{x}{3} + 2y &= 5 \\ \frac{2x - 1}{5} &= y - 1 \end{aligned}\right\}.$$

7. A man engaged to reap a field of corn for 5*s.* an acre; but leaving 6 acres not reaped, he received 2*l.* 10*s.* Of how many acres did the field consist?

8. If $a : b :: c : d$, then $a = b + \frac{(a + b)(c - d)}{c + d}$.

9. Sum the series $5 + 3 + 1 +$ &c. to 13 terms. Of how many terms is the sum $= 0$?

10. If quantities are in geometrical progression, their differences are also in geometrical progression.

IV.

1. Add together $mx^3 - nx^2 - x$, and $px^3 + qx^2 - rx$; and subtract the latter from the former.

2. Multiply $\frac{5}{2}x^2 + 3ax - \frac{7}{3}a^2$ by $2x^2 - ax - \frac{1}{2}a^2$.

Divide $x^4 - \frac{1}{x^4}$ by $x + \frac{1}{x}$.

3. Show that

$$\frac{(a+b)^2}{2b(a-b)} - \frac{a+b}{a-b} = \frac{(a+b)^2}{2b(a-b)} \div \frac{a+b}{a-b}.$$

4. Solve the equations:

(1.) $\frac{x-1}{2} + \frac{x-2}{3} = 2\frac{4}{9} + \frac{x-3}{4}$.

(2.) $(x - 2a)^2 + (x - 2b)^2 = (x - 2)^2 + (x - 2ab)^2$.

(3.) $x^2 + (x - 1)^2 + (x + 2)^2 + (x - 3)^2 = 182$.

5. If each boy in a school spend during the year a number of shillings exceeding that of the boys by 5, the whole pocket-money spent would be 37*l.* 10*s.* What is the number of boys in the school?

6. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$; then $\frac{a+c}{b+d} = \frac{a+e}{b+f} = \frac{c+e}{d+f}$.

7. If two quantities vary as a third, show that their sum, or difference, or the square root of their product, will vary as the third.

8. Find the sum of 9 terms of the series 11, 9, 7, &c.

9. The 9th term of a geometric progression is 27 times as great as the 6th, and the difference of the 3d and 4th terms is 36. Find the series.

10. Find a fourth harmonic proportional to 12, 6, 4.

V.

1. If $a = 7,\ b = 5,\ c = 4,$ show that

$$a^2 + b^2 + c^2 + 2a + 2b + 2c = 13 + \frac{1}{3}\left\{\frac{a^3 - b^3}{a - b} \times (a - c)\right\}.$$

2. Add together

$$x^2 - 3xy - \frac{2}{3}y^2, \quad 2y^2 - \frac{2}{3}y^3 + z^2, \quad xy - \frac{1}{3}y^2 - y^3, \quad \text{an}$$

$$2xy - \frac{1}{3}y^3.$$

3. Multiply together $x - \frac{1}{2},\ x - \frac{3}{2},$ and $x + \frac{1}{2}$.

4. Find the greatest common measure, and the least common multiple of $75a^5b^2c$, and $15a^3bc^4$.

5. Simplify the expression:

$$\frac{a^2 + ab + b^2}{a - b} - \frac{6b^3}{a^2 - b^2} - \frac{a^2 - ab + b^2}{a + b}.$$

6. Solve the following equations:

$$(1.)\quad \frac{1}{2}x + \frac{x + 1}{7} = x - 2.$$

$$(2.)\quad 4x - \frac{12 - x}{x - 3} = 22.$$

$$(3.)\quad \left.\begin{aligned} \frac{2x + 3y}{5} &= 10 - \frac{y}{3} \\ \frac{4y - 3x}{6} &= \frac{3x}{4} + 1 \end{aligned}\right\}.$$

7. A person bought 20 bushels of wheat and 40 bushels of barley for 14*l*., and he observed that $9\frac{1}{2}$ bushels of wheat, and $3\frac{1}{4}$ bushels of barley, cost a fourth of the money. Find the price of a bushel of each.

8. What is meant by a ratio of *greater inequality?* Show that such a ratio is increased by having subtracted from both its terms a quantity less than either of them.

9. A detachment of infantry marched 198 miles in 16 days, and the first day they travelled 18 miles. Supposing each day's march was diminished by the same distance, how far did they travel the last day?

10. The sum of three numbers in geometrical progression is 35, and the mean term is $\frac{2}{3}$ of the difference of the extremes. Find the numbers.

VI.

1. Add together

$$3x^2 + 6xy + y^2, \quad 5x^2 - 2y^2, \quad 3x^3 + 3y^3 - 6x^2 + 1,$$
$$\text{and } x^3 + x^2 + y^3 - y^2.$$

What is the numerical value of the result when $x = 5$, $y = 3$?

2. Develop the continued product

$$(x - a)\ (x + b)\ (x - c)\ (x + d).$$

3. Simplify the expression

$$2a - [3y - \{4x - (5y - 6x - 7y)\}].$$

4. Reduce $\dfrac{x^4 - a^4}{x^2 - 2ax + a^2}$ to its lowest terms, and divide it by $\dfrac{x^2 + ax}{x - a}$.

5. Solve the following equations:

$$(1.)\quad .15x + \frac{.135x - .225}{.6} = \frac{.36}{.2} - \frac{.09x - .18}{.9}.$$

$$(2.)\ \left.\begin{aligned} \frac{x+y}{x-y}+\frac{x-y}{x+y}&=\frac{10}{3} \\ x^2-y^2&=\frac{3}{4} \end{aligned}\right\}.$$

6. How many sheep must a person buy at 7*l.* each, that, after paying one shilling a score for folding them a night, he may gain 79*l.* 16*s.* by selling them at 8*l.*?

7. If $a : b :: c : d$, prove that

$$(a+d)-(b+c)=\frac{(a-b)\ (a-c)}{a}.$$

8. If $y=p+q$, where $p \propto x$ and $q \propto \frac{1}{x}$; also, when $x=1$, $y=6$, and when $x=2$, $y=5$; prove that $y=\frac{4}{3}x+\frac{14}{3x}$.

9. The nth term of an arithmetical progression is $\frac{1}{6}(3n-1)$: show that the sum of n terms is $\frac{1}{12}n(3n+1)$.

10. The 7th term of a geometric series is 48, and the 12th term is 1536. Find the series.

VII.

1. Find the value of the expression

$$\frac{x}{x-a}-\frac{x}{x+a}-\frac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}$$

when $x=2$, $a=1$.

2. Prove that

$$(a+b)^3-(b+c)^3+(c-a)^3=3\ (a+b)\ (b+c)\ (a-c).$$

3. Divide $x^4-9x^2-6xy-y^2$ by x^2+3x+y.

4. Remove the brackets from the expression

$$3a - \{2b - (3b + x)\} + \{b - a - (x - 2b)\}.$$

5. Find the greatest common measure and least common multiple of $x^3 - x$ and $x^3 - 1$.

6. Divide $\dfrac{x^3 + y^3}{x^2 - y^2}$ by $\dfrac{x^2 - xy + y^2}{x - y}$.

7. Square $x^2 - 2$, and add a term to each of the expressions $x^2 - 3ax$, $x^4 - 6x^2$, which shall make them complete squares.

8. Solve the equations:

(1.) $\dfrac{2x-6}{5} - \dfrac{x-4}{9} - \dfrac{3x}{13} = 0.$ (2.) $x^4 - 14x^2 + 13 = 0.$

9. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

10. There are four numbers, the first three of which are in geometrical progression, and the last three in arithmetical; the sum of the first and last is 14, and that of the second and third 12. Find the numbers.

VIII.

1. Find the value of

$$a\sqrt{x^2 - 3a} + x\sqrt{x^2 + 3a},$$

when $x = 5$ and $a = 8$.

2. Add together $ax - by$, $x + y$, and $(a-1)x - (b+1)y$.

3. Divide $x^4 + 64$ by $x^2 + 4x + 8$.

4. Remove the brackets from the expression

$$a - 2(b + 3a) - 3\{b + 2(a - b)\}.$$

5. Show that

$$\frac{1}{(x-2)(x-3)} + \frac{2}{(x-1)(3-x)} + \frac{1}{(x-1)(x-2)} = 0.$$

6. Solve the following equations:

$$(1.)\ \frac{3x-2}{4} - \frac{6x-5}{8} = \frac{5x}{16}.$$

$$(2.)\ \left.\begin{aligned} \frac{x}{3} + \frac{y}{2} &= \frac{4}{3} \\ \frac{x}{2} + \frac{y}{3} &= \frac{7}{6} \end{aligned}\right\}.$$

$$(3.)\ 12x^2 - 23x = 242.$$

7. The square of three less than the fifth part of a troop of monkeys had gone to a cave, and one monkey was in sight, having climbed on a branch. Say how many there were.

8. If $a : b :: c : d$, prove that

$$a^2 : b^2 :: a^2 + c^2 : b^2 + d^2.$$

9. Supposing that the space through which a body falls from rest in any time varies as the square of the time, and that in 3 seconds it falls through 144 feet; through how many feet will it fall in 4 seconds?

10. Four men share a sovereign between them, their shares being in arithmetical progression, and the lowest share is two shillings. What are the shares of the other three men?

IX.

1. Find the value of

$$(\sqrt{x^2 + y^2} + z)\ (\sqrt{x^2 + y^2} - z),$$

when $x = 4$, $y = 5$, $z = 6$.

2. Distinguish between the addition of algebraical and that of arithmetical quantities.

Add together $16a^2 - 7ab - 8b^2 + 3c$, $4b^2 - 8c + ab$, and $12ab - 8a^2 + 5c$.

3. Prove that $(y-z)^3+z^3-y^3=3yz(z-y)$.

4. Prove that

$$\frac{a+b}{a^2+ab+b^2}-\frac{a-b}{a^2-ab+b^2}=\frac{2b^3}{a^4+a^2b^2+b^4}.$$

5. Solve the following equations:

(1.) $x+\frac{3x-1}{5}=\frac{2x+5}{3}$.

(2.) $\frac{x-\frac{1}{x}}{x+\frac{1}{x}}=\frac{3}{5}$.

(3.) $\left.\begin{aligned} x+y-6&=0 \\ 4x^2-7y^2+27&=0 \end{aligned}\right\}$.

6. A person rows in 4 hours a distance of 32 miles down a river which runs uniformly at the rate of 3 miles an hour. How long will he take to return at the same rate of exertion?

7. Find the mean proportional between $\frac{7}{9}$ and $3\frac{4}{7}$.

8. Given that the volume of a sphere varies as the cube of the diameter, and that a sphere of 14 inches in diameter has a volume of 1433·74 cubic inches. Find the number of cubic feet in a sphere 2 feet in diameter.

9. Find the sum of 15 terms of a series in arithmetical progression of which the 8th term is $\frac{4}{9}$.

10. If a, b, c, be in arithmetical progression, and a, mb, c, in geometrical progression; then a, m^2b, c, are in harmonical progression.

X.

1. If $a=2$, $b=5$, $c=3$, $s=10$, show that

$$s(s-2a)(s-2b)(s-2c)=4a^2b^2-(a^2+b^2+c^2)^2.$$

2. Add together

$$x^3-4x^2+9x-10 \text{ and } x^3+2x^2-3x+20,$$

and divide the result by x^2-2x+5.

3. Multiply $a^3 + 3a^2b - 2ab^2 + 3b^3$ by $a^2 + 2ab - 3b^2$, and divide $a^4 - 16b^4$ by $a - 2b$.

4. Find the greatest common measure and the least common multiple of $6ax^3$, $9a^2x$, and $24x^5$.

5. Bracket the terms of the following expressions *two* together:

(1.) $a - b - c - d + e - f$,

(2.) $-3a + 2b - c + 3d + e - 4f$.

Also, collect the terms in brackets *three* together, placing the two last terms in each bracket under a vinculum.

6. Simplify the following expressions:

(1.) $\dfrac{x^4 - 4y^4}{x^4 + 4y^4} \div \dfrac{x^2 - y^2}{x^2 - 2xy + 2y^2}$.

(2.) $\dfrac{1}{x - 1 + \dfrac{1}{1 + \dfrac{x}{4 - x}}}$.

7. Solve the equations:

(1.) $(x - a)(x + b) + a^2 = (x + a)(x - b) + b^2$.

(2.) $\dfrac{x + 1}{x} + \dfrac{x}{x + 1} = \dfrac{13}{6}$.

8. A number of individuals propose to build a church costing 6,000*l.* by equal payments: before it is begun four become bankrupts and six others refuse to augment their subscriptions, so that each of the rest has to pay 8*l.* more. Find the number of individuals.

9. Find two numbers in the ratio of 3 : 4, and of which the sum : the sum of their squares :: 7 : 50.

10. The difference between the sums of m and n terms of an arithmetical progression : the sum of $m + n$ terms :: $m - n : m + n$.

XI.

1. Add together $3a^2+3ax-x^2$, $a^2-ax+2x^2$, $-2x^2-ax$, and $-(3a^2+ax.)$ What is the result when $x=a$?

2. Subtract $-(a+x)$ from $a-x$.

Simplify

$\{2a^2-(2b^2+c^2)\}-(3b^2+2a^2-c^2)-(3c^2-2b^2-a^2)$.

3. Find the continued product of $x-3$, $x+1$, $x-1$, $x+3$; and divide the result by x^2+2x-3.

4. Add together $\dfrac{x(a+x)}{a-x}$, $\dfrac{5ax-x^2}{x-a}$, and $\dfrac{2a^2}{a-x}$.

Prove that $1-\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{(c+a-b)(c-a+b)}{2ab}$.

5. Solve the following equations:

(1.) $\dfrac{10x+17}{18}-\dfrac{12x+2}{13x-16}=\dfrac{5x-4}{9}$.

(2.) $2x^2-7x+6=0$.

(3.) $(3x-2)(x-1)=14$.

(4.) $\left.\begin{aligned}\frac{5x}{2}+\frac{3y}{4}&=7\\ \frac{5y}{4}-\frac{7x}{6}&=11\end{aligned}\right\}$.

6. In going 210 yards the fore-wheel of a carriage makes 7 revolutions more than the hind-wheel; but if the circumference of each wheel be increased 1 yard, it will make only 5 revolutions more than the hind-wheel in going the same distance. Find the circumference of each wheel.

7. What quantity added to both terms of $a:b$ will make it the ratio $c:d$?

8. If $xy \propto x^2-y^2$, and when $x=3$, $y=2$; show that when $y=3$, x is $4\frac{1}{2}$.

9. The 7th term of an arithmetic series is 20, and the 20th term is 7. Find the series.

10. The difference between the first and second of four numbers in geometrical progression is 36, and the difference between the third and fourth is 4. What are the numbers?

XII.

1. Find the value of

$$\frac{x(x-1)a^3 + (x^3 + 2x - 2)a^2 + (3x^2 - x^3)a - x^4}{a^2x + 2a - x^2},$$

when $x = 2$, and $a = 3$.

2. Prove that

$$(a-b)^3 + (b-c)^3 + (c-a)^3 = 3(a-b)(b-c)(c-a).$$

3. Divide $6y^3 + 14y^2 - 4y + 24$ by $2y + 6$.

4. Simplify the expressions,

$$(a+b) - (2a - 3b) - (5a + 6b) - (-7a + b),$$

$$\frac{(a-1)^2}{a^2(a-b)} + \frac{a+b}{a^2b^2} + \frac{(b-1)^2}{b^2(b-a)}.$$

5. Solve the equations:

$$(1.)\ \frac{x+4}{x-4} + \frac{x+2}{x-2} = 7. \qquad (2.)\ \left.\begin{array}{l} x - 4y = 7 \\ \dfrac{x}{3y} + \dfrac{33}{10} = \dfrac{4x-y}{5y} \end{array}\right\}.$$

6. A person buys a piece of land at 30*l.* an acre, and by selling it in allotments finds the value increased threefold, so that he clears 150*l.*, and reserves 25 acres for himself. How many acres were there?

7. When is a quantity said to vary directly as another, and when as two others jointly? Exemplify in solving the question: If a garrison of 600 men have provisions for 5 weeks, allowing each man 12 oz. a day; how many men can be maintained 10 weeks by the same quantity, if each man is limited to 8 oz. a day?

8. There are two numbers in the ratio of $\frac{1}{2}$ to $\frac{2}{3}$, which being increased respectively by 6 and 5, are in the ratio of $\frac{2}{5}$ to $\frac{1}{2}$. Required the numbers.

9. If s be the sum of n terms of an arithmetic series, of which a and c are the first and third terms; show that

$$\frac{4s}{n} = (n-1)\,c - (n-5)\,a.$$

10. If A and G be the arithmetic and geometric means respectively between two quantities, A' the arithmetic mean between their squares; show that

$$A' = 2A^2 - G^2.$$

XIII.

1. Find the value of

$$20ab - 7bc + 16ac - 5d^2,$$

when a, b, c, d are equal to 1, 3, 4, 5 respectively.

2. Multiply $a^2 + 2ab + b^2 - c^2$ by $a^2 - 2ab + b^2 + c^2$; and show that the result may be expressed under the form

$$(a^2 - b^2)^2 - c^2(c^2 - 4ab).$$

3. Divide $x + y$ by $\frac{1}{x} + \frac{1}{y}$.

4. Prove that $\dfrac{a^2 + ab + b^2}{a+b} - \dfrac{a^2 - ab + b^2}{a-b} = \dfrac{2b^3}{b^2 - a^2}$.

5. Reduce to their simplest forms the expressions,

$$\left(\frac{1}{a} + \frac{a^2}{b^3}\right) \div \left(\frac{1}{a} - \frac{1}{b} + \frac{a}{b^2}\right), \qquad \frac{(a+b)^2 - (c+d)^2}{(a+d)^2 - (b+c)^2}.$$

6. Solve the following equations:

$$(1.)\ \frac{1}{x-3} + \frac{1}{x-6} = \frac{2}{x-9}.$$

(2.) $\left.\begin{array}{l} x - y = 3 \\ x^2 - y^2 = 15 \end{array}\right\}$. (3.) $4(x-1) - \frac{(x-1)}{2x} = 3\frac{3}{4}$.

7. A lady drinks a pint of wine and water, while a gentleman drinks three pints of wine; supposing the gentleman to have drunk five times as much wine as the lady, find how much water there was in the lady's pint.

8. Supposing that three Englishmen, four Frenchmen, and five Turks are together a match for sixteen Russians, and that four Englishmen, six Frenchmen, and ten Turks are together a match for twenty-five Russians; prove that two Englishmen and two Frenchmen are together a match for seven Russians.

9. One horse takes 6 strides whilst another takes 5, but 7 strides of the latter horse are equal to 8 strides of the former; which is the swifter horse?

10. Insert eighteen harmonic means between 1 and $\frac{1}{20}$.

XIV.

1. Find the sum of the fractions,

$$\frac{a}{b+c-a}, \quad \frac{b}{c+a-b}, \quad \frac{c}{a+b-c},$$

when $a = 3$, $b = 4$, $c = 5$.

2. Multiply $x^3 + x^2y + xy^2 + y^3$ by $x - y$, and find the quotient of $x^3 + (a+b+c)x^2 + (bc+ca+ab)x + abc$ divided by $x + a$.

3. Remove the brackets from the expression,

$$2(a+b) - [3(c-d) + \{a+b-4(c-d)\}].$$

4. Add together,

$$\frac{1}{3a(x-a)} \text{ and } \frac{a-x}{3a(x^2+ax+a^2)}.$$

5. Reduce to lowest terms

$$\frac{(x^3+y^3)(x^2+y^2)(x+y)}{(x^4-y^4)(x^2-xy+y^2)}.$$

6. Solve the equations:

(1.) $\frac{x+1}{3}-\frac{x-1}{4}=\frac{x-2}{5}-\frac{x-3}{6}+\frac{31}{30}.$

(2.) $\left.\begin{array}{l}\frac{x^2}{y^2}-\frac{3}{4}\cdot\frac{x}{y}=\frac{7}{9}\\ x+y=7\end{array}\right\}.$

7. A is 30 years younger than B, and 5 years back B was twice the age of A. What is A's age?

8. If $a : b :: c : d :: e : f$, then

$$a : b :: ma+nc+pe : mb+nd+pf.$$

9. Given that the volume of a sphere varies as the cube of its radius. Prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are 3, 4, 5 inches.

10. There are four numbers of which the first three are in arithmetical, and the last three in harmonical, progression. Show that these numbers, taken in order, form a proportion.

XV.

1. Multiply x^3-3x+2 by x^3-3x-2, and divide the product by x^2-1.

2. Simplify $\frac{x-a}{b}-\frac{x-b}{a}$, when $x=\frac{a^2}{a-b}$; and show that

$$(x^2+xy+y^2)^2-(x^2-xy+y^2)^2=4xy(x^2+y^2).$$

3. From $\frac{2}{x-1}$ subtract $\frac{1}{x+1}+\frac{x}{x^2+1}+\frac{3}{x^2+1}.$

4. Simplify the expressions,

$$\frac{2x}{x^2-y^2}-\frac{1}{x+y},\qquad \frac{a^2+ab+b^2}{a^3-b^3}+\frac{a^2-ab+b^2}{a^3+b^3}.$$

5. Solve the following equations:

(1.) $\left(x - \frac{5}{2}\right) \left(x + \frac{3}{2}\right) - (x - 5)(x + 3) - 9\frac{3}{4} = 0.$

(2.) $\left.\begin{array}{l} x^2(x + y) = 80 \\ x^2(2x - 3y) = 80 \end{array}\right\}.$

(3.) $(x + 1)(2x + 3) = 4x^2 - 22.$

6. A and B play at a game, agreeing that the loser shall always pay to the winner one shilling more than half the money the loser has; they commence with equal quantities of money; but after B has lost the first game, and won the second, he has twice as much as A. How much had each at the commencement?

7. Two vessels A and B contain each a mixture of water and wine, A in the ratio of $3 : 4$, B in that of $5 : 6$. What quantity must be taken from each to form a mixture which shall consist of 7 gallons of water and 11 of wine?

8. Two quantities whose sum is y vary, the one as x, and the other as $\frac{1}{x}$, and when $x = a, b$, $y = p, q$, respectively; express y in terms of x.

9. The sum of an arithmetical progression whose first term is 2, and last term 42, is 198. Find the common difference and the number of terms.

10. If a, b, c be supposed to be first in geometrical progression and next in harmonical progression, the ratio $a : b$ in the first case will be equal to the ratio $a : c$ in the second.

XVI.

1. Multiply $a^2 + 2ab + b^2 - c^2$ by $a^2 - 2ab + b^2 + c^2$, and show that the result may be put under the form

$$(a^2 - b^2)^2 + c^2(4ab - c^2).$$

2. Show that $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$.

3. Find the continued product of $x - a$, $x - b$, and $x - c$, and in the result put $x = a + b + c$, and reduce the expression to its simplest form.

4. Simplify the expressions,

$$\frac{x+1}{x+5} - \frac{x+5}{x+1} + \frac{1}{(x+1)\,(x+5)}.$$

5. Show that

$$\frac{1}{(4-x)\,(5-x)} - \frac{2}{(4-x)\,(6-x)} + \frac{1}{(5-x)\,(6-x)} = 0.$$

6. Solve the equations:

$$(1.)\ \frac{1}{7}\left(x - \frac{1}{2}\right) - \frac{1}{5}\left(\frac{2}{3} - x\right) = 1\,\frac{13}{30}.$$

$$(2.)\ \frac{2x+1}{2x-1} + \frac{3x-2}{3x+2} = \frac{13}{6}.$$

$$(3.)\ \left.\begin{aligned} \frac{x}{8} + \frac{y}{9} &= 16 \\ \frac{x}{9} + \frac{y}{8} &= 15\,\frac{7}{8} \end{aligned}\right\}.$$

7. Two masons, A and B, were engaged in hewing columns; B worked steadily and finished his column in 6 days; A being a much quicker workman did not seriously put forth his powers till the fourth day, doing on the first day only one-sixth, on the second one-third, and on the third one-half of that which B did. A had completed his work one day before B. Compare their rates of work.

8. Divide a into four parts, such that the first increased by b, the second diminished by b, the third multiplied by b, and the fourth divided by b, may be all equal.

9. If $a : b :: c : d$, then

$$a : b :: \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}.$$

10. If there be 5 quantities a_1, a_2, a_3, a_4, a_5, such that a_1, a_2, a_3, are in arithmetical progression; a_2, a_3, a_4, in geometrical progression; and a_3, a_4, a_5, in harmonical progression; then a_1, a_3, a_5 will be in geometrical progression.

THE END.

APPENDIX II.

MEASURES AND MULTIPLES.

1. Show that if one quantity measure another it will measure any multiple of that quantity; and if a quantity measure two others it will measure their sum or difference.

2. State and prove the rule for finding the greatest common measure of two numbers, the prime factors of which are not obvious.

3. In applying this rule to find the greatest common measure of two compound algebraical quantities, what modifications are necessary? Illustrate by finding the greatest common measure of $a^2 - 5ab + 4b^2$ and $a^3 - a^2 b + 3ab^2 - 3b^3$.

4. Find the greatest common measure of

(1.) $x^3 - x^2 - 8x + 12$, and $3x^2 - 2x - 8$.

(2.) $x^3 + 4x^2 - 5$, and $x^3 - 3x + 2$.

(3.) $x^3 - 19x^2 + 119x - 245$, and $3x^2 - 38x + 119$.

(4.) $7x^2 - 12x + 5$, and $2x^3 + x^2 - 8x + 5$.

(5.) $a^3 + a^2 b + ab + b^2$, and $a^4 - b^4$.

(6.) $x^8 + a^2 x^6 + ax^2 + a^3$, and $x^6 - a^4 x^2 - ax^4 + a^5$.

(7.) $12x^4 - 88x^2 - 60x$, and $10x^4 - 34x^3 + 36x$.

(8.) $20x^4 + x^2 - 1$, and $25x^4 + 5x^3 - x - 1$.

5. Reduce to lowest terms the following fractions:

(1.) $\dfrac{x^3 - 6x^2 + 11x - 6}{x^3 + 4x^2 + x - 6}$. (4.) $\dfrac{x^3 - 4x^2 + 9x - 10}{x^3 + 2x^2 - 3x + 20}$.

(2.) $\dfrac{9x^3 - 6x^2 - 3x}{6x^2 - x - 1}$. (5.) $\dfrac{x^2 - (a - b)x - ab}{x^2 - (a + b)x + ab}$.

(3.) $\dfrac{x^3 + 1}{x^3 + mx^2 + mx + 1}$. (6.) $\dfrac{3x^3 - 3ax^2 + a^2x - a^3}{4x^2 - ax - 3a^2}$.

6. Prove that the least common multiple of two quantities is equal to their product divided by their greatest common measure.

7. Show that every multiple of two or more quantities is a multiple of their least common multiple.

8. Find the least common multiple of

(1.) $x^3 + y^3$, and $(x + y)^2$.

(2.) $x^4 - 1$, and $x^3 + 3x^2 - 4$.

(3.) $x^3 - 11x^2 + 32x - 28$, and $3x^2 - 22x + 32$.

(4.) $x^3 - 4x^2 + 9x - 10$, and $x^3 + 2x^2 - 3x + 20$.

(5.) $12x^2 - 17ax + 6a^2$, and $9x^2 + 6ax - 8a^2$.

(6.) $x^3 + x^6y^2 + x^2y + y^3$, and $x^4 - y^4$.

(7.) $x^2 + 2x - 3$, $x^2 + 5x + 6$, and $2x^2 - x - 10$.

(8.) $x^2 - 9x + 20$, $x^2 + 6x - 55$, and $x^2 - 25$.

(9.) $2x^2 + 3x + 1$, $x^2 - x - 2$, $2x^2 - x - 1$, and $x^2 - 5x + 6$.

INDICES, SQUARE ROOT, AND SURDS.

1. If m, n be whole numbers, show that $(a^m)^n = a^{mn}$.

2. If the laws of indices, viz. $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(a^m)^n = a^{mn}$ be supposed true *whatever* the values of m and n, show that we must have

$$a^0 = 1,\ a^{-m} = \frac{1}{a^m},\ \text{and } a^{\frac{p}{q}} = \text{the } q^{\text{th}} \text{ root of } a^p.$$

3. Simplify the following quantities:
$a^2 \times a^{-3} \times a^4$, $(a^2b^3)^4$, $(3a^6b^4c^3)^3$, $ab^2c^3 \div a^{-3}c$, $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$, $ab^{\frac{1}{2}} \times cb^{-\frac{1}{3}}$, $x^{\frac{1}{2}} \times x^{\frac{1}{4}} \times x^{\frac{1}{16}} \div x^3$, $2a^{-2} \times 3a^{-3} \times 4a^{-5} \times 5a^6$, $-3a^{-2}b^2c^{-1} \times 5a^{-4}b^{-6}c^3$, $\dfrac{18a^{-5}b^3}{7c^{-2}d^{-6}} \times \dfrac{4a^6b^{-5}}{9c^3d^9}$.

4. Show that $(-a)^m$ is equal to a^m or $-a^m$ according as m is an even or odd integer.
Simplify $(-ab)^5 \times a^{-3}b^{-2}c^4$. $\{-(-a)^4\}^3$.

5. Investigate a rule for extracting the square root of a compound algebraical quantity by considering how the root $a+b$ can be derived from its square $a^2 + 2ab + b^2$.

6. Find the square roots of the following:

(1.) $x^2 + 4xy + 4y^2$.

(2.) $9a^2 + 12ab + 4b^2$.

(3.) $x^4 - 4ax^3 + 6a^2x^2 - 4a^3x + a^4$.

(4.) $x^2 - ax + \dfrac{a^2}{4}$.

(5.) $9x^4 + 12x^3 + 10x^2 + 4x + 1$.

(6.) $4a^2x^4 - 12a^3x^3 + 13a^4x^2 - 6a^5x + a^6$.

(7.) $9 - 24x - 68x^2 + 112x^3 + 196x^4$.

(8.) $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

(9.) $1 - 2x + 3x^2 - 4x^3 + 3x^4 - 2x^5 + x^6$.

(10.) $1 + x$ to five terms.

(11.) $a^2 + x^2$ to four terms.

(12.) $\frac{a^2}{b^2} + 2$ to four terms.

7. Show that the cube of $a + b$ is $a^3 + 3a^2b + 3ab^2 + b^3$; and investigate a rule for finding cube roots by considering how the former of these quantities may be obtained from the latter.

8. Find the cube roots of

(1.) $x^3 - 3x^2y + 3xy^2 - y^3$.

(2.) $x^3 - 6x^2y + 12xy^2 - 8y^3$.

(3.) $8a^3 - 84a^2x + 294ax^2 - 343x^3$.

(4.) $27x^3 - 54x^2y + 36xy^2 - 8y^3$.

(5.) $1 + 3x - 5x^3 + 3x^5 - x^6$.

(6.) $a^6 - 6a^5 + 40a^3 - 96a - 64$.

(7.) $a^3 + x^3$ to three terms.

9. Explain how the rules for square and cube roots are applied to numbers.

Find the cube roots of 2197, 12167, 1367631.

10. What is meant by a "surd?" Define *similar* surds, giving examples.

11. Prove that $\sqrt{a}\sqrt{b} = \sqrt{ab}$; $\sqrt{a} \div \sqrt{b} = \sqrt{\frac{a}{b}}$;

$$\sqrt[m]{a}\,\sqrt[n]{a} = \sqrt[mn]{a^{m+n}}; \quad \sqrt[m]{a} \div \sqrt[n]{a} = \sqrt[mn]{a^{n-m}};$$

$$\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[mn]{a^n b^m}; \quad \sqrt[m]{a} \div \sqrt[n]{b} = \sqrt[mn]{a^n b^{-m}}.$$

12. Simplify the following surds:

(1.) $\sqrt{50}$, $\sqrt{128}$, $\sqrt{242}$.

(2.) $\sqrt{243}$, $\sqrt{125}$, $\sqrt{96}$.

(3.) $\sqrt{16}$, $\sqrt[3]{81}$, $\sqrt[3]{686}$.

(4.) $\sqrt[4]{a^4b}$, $\sqrt[5]{x^6y^5}$, $\sqrt[n]{a^{2n}b^{3n}}$, $\sqrt[3]{-27a}$.

(5.) $\sqrt{4^3} + \sqrt[3]{8^5}$, $\sqrt{8} + \sqrt{98} - \sqrt{18}$.

(6.) $\sqrt{x^2y} + \sqrt{xy^2} - (x\sqrt{y} - y\sqrt{x})$.

(7.) $\sqrt{a^2c} + \sqrt{b^2c} + \sqrt{c^3}$, $\sqrt{4a^2b + 4a^2x}$.

(8.) $\sqrt[3]{54} + 3\sqrt[3]{16} - 2\sqrt[3]{2}$, $\sqrt{\frac{8}{27}} - \sqrt{\frac{1}{6}}$.

(9.) $\sqrt{3} \times \sqrt{5}$, $3\sqrt{2} \times 5\sqrt{2}$, $2\sqrt{2} \times 3\sqrt{3} \times 4\sqrt{4}$.

(10.) $2\sqrt[3]{4} \times 3\sqrt[3]{2}$, $\sqrt[3]{2} \times \sqrt[3]{3^2} \times \sqrt[3]{12}$.

(11.) $\sqrt{xy^3} \times \sqrt{x^3y}$, $\sqrt{xy} \times \sqrt{xz} \times \sqrt{yz}$.

(12.) $\sqrt[3]{a^2b} \times \sqrt[3]{ab^2}$, $\sqrt{a} \times \sqrt[3]{a}$.

(13.) $x\sqrt{\frac{y}{x}} + y\sqrt{\frac{x}{y}}$, $\sqrt[3]{a} \times \sqrt[5]{a^2}$.

(14.) $abc\sqrt{a^2b^3} \times \sqrt[3]{bc^3} \times b^{-\frac{1}{3}}$.

(15.) $\sqrt{a+b} \times \sqrt{a-b}$, $\sqrt{(\sqrt{5}+1)(\sqrt{5}-1)}$.

(16.) $(\sqrt{x+a} - \sqrt{a})(\sqrt{x+a} + \sqrt{a})$.

(17.) $\sqrt{32} \div \sqrt{12}$, $\sqrt[4]{a^5b} \div \sqrt[4]{ab^3}$.

(18.) $\sqrt{a^3} \div \sqrt[3]{a^2}$, $ab \div \sqrt[5]{a^2b^2}$.

(19.) $\sqrt{ax + x^2} \div \sqrt{ax}$, $\sqrt{x^2 - a^2} \div \sqrt{x+a}$.

(20.) $(a + 2\sqrt{ab} + b) \div (\sqrt{a} + \sqrt{b})$.

13. Square $1 + \sqrt{x^2 - 1}$; show that the result, multiplied by $x^2 - 2\sqrt{x^2-1}$ is equal to $(x^2 - 2)^2$.

14. Simplify $(x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)$.

15. Multiply $a^{\frac{5}{3}} + a^{2}b^{\frac{1}{3}} + a^{\frac{2}{3}}b^{\frac{2}{3}} + ab + a^{\frac{1}{3}}b^{\frac{4}{3}} + b^{\frac{5}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

16. Divide $a^3 - b^3$ by $a^{\frac{3}{4}} - b^{\frac{3}{4}}$.

17. Square and cube $a + b\sqrt{x}$.

MISCELLANEOUS EXAMPLES.

SECOND SERIES.

I.

1. MULTIPLY together ax^{m+2}, bx^n, cx^{p-2}. Divide the result by x^{m+n-p}.

2. Divide $1 - x$ by $1 - x + x^2$ to four terms.

3. Simplify the expressions:

$$\frac{2a(x^2 - y^2)^2}{cx} \times \frac{x^3}{(x-y)(x+y)^2};$$

$$(x - ae)^2 + (1 - e^2)(a^2 - x^2).$$

4. What are the factors of $x^3 + y^3$, and of $x^3 - y^3$? Show that the product of these quantities is divisible by $x^4 + x^2y^2 + y^4$.

5. Solve the following equations:

$$(1.)\ \frac{a}{x-a} + \frac{b}{x-b} = \frac{2c}{x-c}; \qquad (2.)\ \left.\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 1 \\ \frac{x}{3} + \frac{y}{4} &= \tfrac{1}{2} \end{aligned}\right\};$$

$$(3.)\ \sqrt{x} + 2 = \frac{3}{\sqrt{x}}.$$

6. The length of a field is twice its breadth, and another field which is 50 yards longer and 10 yards broader contains 6800 square yards more than the former: find the size of either.

7. Show that the ratio $x^3 + y^3 : x^2 + y^2$ is greater than $x^2 + y^2 : x + y$.

8. If $x + y \propto x - y$, show that $x^2 + y^2 \propto xy$.

9. Show that when u, r, v are in harmonical progression,

$$\left(u - \frac{r}{2}\right)\left(v - \frac{r}{2}\right) = \left(\frac{r}{2}\right)^2.$$

10. The differences of three quantities in geometrical progression are in arithmetical progression. Find the common ratio.

II.

1. If 1 be divided into any two parts, prove that the sums, formed by adding each part to the square of the other, are equal.

2. What is the continued product of $a^2 + ab + b^2$, $a - b$, $a^2 - ab + b^2$, and $a + b$?

3. Find the least common multiple of

$$x + y, x - y, \text{ and } x^2 - y^2;$$

and the greatest common measure of

$$2x^3 - 5x^2 + 7x - 6, \text{ and } 2x^3 - x^2 + 3x - 9.$$

4. Simplify

$$\frac{1}{6}\left\{\frac{1}{x-1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} - \frac{x+2}{x^2+x+1}\right\}.$$

5. Solve the equations:

(1.) $\dfrac{x+1}{x-1} + \dfrac{x+2}{x-2} = 2 \cdot \dfrac{x+3}{x-3}$; (2.) $\left.\begin{aligned} x+2y+3\frac{y}{x} &= 16 \\ 3x+y+3\frac{x}{y} &= 23 \end{aligned}\right\}$;

(3.) $a + x + \sqrt{a^2 + x^2} = b$.

6. Show that if $y = ax + \dfrac{m}{a}$, and $y + \dfrac{x}{a} + ma = 0$,

$$\text{then } x + m = 0.$$

7. There are three casks containing respectively a, b, and c gallons of gin, rum, and brandy; the a gallons out of the first cask are divided equally between the second and third, and half the contents of each of these casks is then poured into the first: determine the proportions of the different spirits in this last mixture.

8. If $\frac{1}{x} + \frac{1}{y} \propto z$, $\frac{1}{z} + \frac{1}{x} \propto y$, and $\frac{1}{y} + \frac{1}{z} \propto x$,

then $x + y + z \propto xyz$.

9. What limit does r^n tend to, when r is a proper fraction, and n increases indefinitely?

Show that $a + ar + ar^2 + \dots$ ad inf. $= \frac{a}{1-r}$ when r is less than 1.

10. If P, Q, R be the p^{th}, q^{th}, and r^{th} terms of a harmonic series; then

$$\frac{P-Q}{p-q} \cdot R = \frac{R-P}{r-p} \cdot Q = \frac{Q-R}{q-r} \cdot P.$$

III.

1. Given $\mu = 1{\cdot}5$, $f = -20$, $y = 2$, $r = -20$, $s = 20$, find the value of

$$-\frac{\mu - 1}{\mu^2}\left\{\frac{1}{r^3} - \left(\frac{1}{s} - \frac{1}{f}\right)^2 \left(\frac{1}{s} - \frac{\mu + 1}{f}\right)\right\}\frac{f^2 y^2}{2}$$

2. Show that

$$a^2 - ax + x^2 - \frac{2x^3}{a+x} = \frac{a^3 - x^3}{a+x}.$$

3. Resolve into factors $6x^2 - 5x + 1$, $6x^2 + 5x + 1$, $6x^2 + x - 1$, and $6x^2 - x + 1$; and find the square root of $x^4 - 6x^3 + 17x^2 - 24x + 16$.

4. Show that

$$\frac{1}{4(x+1)} - \frac{1}{4} \cdot \frac{x-1}{x^2+1} - \frac{1}{2} \cdot \frac{x-1}{(x^2+1)^2} = \frac{1}{(x^2+1)^2(x+1)};$$

and reduce $\frac{x^2 + y^2 - z^2 - 2xy}{x^2 + y^2 - z^2 + 2yz}$ to lowest terms.

5. If $a^2e^2 = a^2 + b^2$; then

$$\frac{be+a}{a} - \frac{b}{b+ae} = e^2.$$

6. Solve the equations:

(1.) $\frac{x}{2} + \frac{2}{x} = \frac{x}{3} + \frac{3}{x}$. (2.) $\left.\begin{array}{l} 1 + x = y \\ m(1 - x^2) = ny^2 \end{array}\right\}$.

(3.) $\sqrt{1+x} + \sqrt{1-x} = \sqrt{2}$.

7. Nine numbers are arranged in a square, and are such that the sum of those in every straight row of three is m. Show that the centre figure $= \frac{m}{3}$, and that the four corner numbers amount to $\frac{4m}{3}$.

8. If $\frac{l}{a-b} = \frac{m}{b-c} = \frac{n}{c-a}$; then $l + m + n = 0$.

9. If $x + y \propto z$ and $x + z \propto y$; then $y + z \propto x$.

10. If a be the first term, l the n^{th}, s the sum of n terms, σ the sum *ad infinitum* of a geometric series of which the common ratio is less than 1, prove that

$$\frac{l}{a} + \frac{s-l}{\sigma} = 1.$$

IV.

1. If $\{(x+a)^2 + y^2\}\{(x-a)^2 + y^2\} = a^4$;
then $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$.

2. Resolve $x^4 - a^4$, $x^6 - a^6$ into their simple factors.

3. Find the greatest common measure and the least common multiple of $(x - y)^3$ and $(x^2 - y^2)^2$.

4. Reduce the following expressions:

$$\frac{3}{2(x+3)} + \frac{2}{x+2} - \frac{1}{2(x+1)}; \quad \frac{\dfrac{x-1}{x^2+x+1} - \dfrac{x+1}{x^2-x+1}}{\dfrac{x^2+1}{x^6-1} - \dfrac{2}{3(x^2-1)}}.$$

$$\sqrt{9xy} + \sqrt{289xy} - 2\sqrt{2}\sqrt{50xy}.$$

5. Solve the equations:

(1.) $\dfrac{5x-6}{9} - \dfrac{2x-13}{3} = \dfrac{x+7}{3}$. (2.) $\left.\begin{array}{l} \dfrac{m}{x} + \dfrac{n}{y} = a \\ \dfrac{n}{x} + \dfrac{m}{y} = b \end{array}\right\}$.

(3.) $(x^2 + 1)(x + 2) = 2$.

6. If $x_1^2 + px_1 + q = 0 = x_2^2 + px_2 + q$;

then $x_1 + x_2 = -p$, and $x_1 x_2 = q$.

7. Show that each of the series of equal ratios $a_1 : b_1$, $a_2 : b_2$, $a_3 : b_3$, &c.

$$= \frac{n_1 a_1 + n_2 a_2 + n_3 a_3 + \&c.}{n_1 b_1 + n_2 b_2 + n_3 a_3 + \&c.}$$

8. The value of glass mirrors varies as the cube of the size. Show that by cutting a plate of 10 square feet into four of 1, 2, 3, 4, square feet, nine-tenths of the value is sacrificed.

9. If the second term of an arithmetical progression be a mean proportional between the first and fourth, show that the sixth term will be a mean proportional between the fourth and the ninth.

10. If a, b, c, d be in geometrical progression,

$$(a + b + c + d)^2 = (a + b)^2 + (c + d)^2 + 2(b + c)^2.$$

V.

1. Show that the rule for division of powers leads to the symbolical equality $a^0 = 1$.

$$\text{Show also that } a^{-m} = \frac{1}{a^m}.$$

2. *Prove* that $a - (b + c) = a - b - c$,
and $a - (b - c) = a - b + c$.

3. Show that

$$\frac{a^2 - (b - c)^2}{(a + c)^2 - b^2} + \frac{b^2 - (a - c)^2}{(a + b)^2 - c^2} + \frac{c^2 - (a - b)^2}{(b + c)^2 - a^2} = 1.$$

4. What are the factors of $x^4 + x^2y^2 + y^4$? Is $x^4 - x^2y^2 + y^4$ composite?

5. Find the least common multiple of

$$x^2 - 3x + 2,\ x^2 - x - 2,\ x^2 + x - 2,$$

and of $x^3 - 2x^2 + 1$, and $x^3 - 2x - 1$.

6. Given $\dfrac{a^3 + x^3}{a + x} + \dfrac{a^3 - x^3}{a - x} = 4a^2$; find x.

7. If $y = mx + \sqrt{m^2a^2 + b^2}$, and $my + x = 0$;
then $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

8. The average importation of wheat from Ireland during the last $m + n$ years was a quarters, and during the last n years was b quarters: what was the average annual importation during the first m years of the former period?

9. If S_a represent the sum of n of the natural numbers beginning with a, prove that

$$S_{3a+n-1} = 3\,S_a.$$

10. Sum the following infinite series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \&c. \ldots .$$

$$1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \&c. \ldots .$$

$$1 + \frac{x}{x+1} + \frac{x^2}{(x+1)^2} + \&c. \ldots .$$

VI.

1. Prove that $(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2$
$= (a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - (ax + by + cz)^2$.

2. Find the value of

$$x + \frac{1}{1-x} - \frac{1-x}{x} - a - \frac{1}{1-a} + \frac{1-a}{a}$$

when $ax = 1$.

3. Express $\frac{2}{1-x}$, $\frac{a^2}{x^2 - xy + y^2}$, $\frac{\sqrt{a-x}}{\sqrt{a+x}}$, $\frac{\sqrt{x+y}}{\sqrt{x-y}}$, $\frac{1}{1+\sqrt{1-x^2}}$, with denominators $1 - x^2$, $x^3 + y^3$, $a + x$, $\sqrt{x^2 - y^2}$, x^2, respectively.

4. Prove that

$$1 + \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} = \frac{(a+b)^2 - (c-d)^2}{2(ab+cd)};$$

and

$$1 - \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} = \frac{(c+d)^2 - (a-b)^2}{2(ab+cd)}.$$

5. Find the least common multiple of

$$3x^2 - 5x + 2 \text{ and } 4x^3 - 4x^2 - x + 1.$$

6. Solve the equations:

(1.) $m\left(\frac{x+a}{x+b}\right)+n\left(\frac{x+b}{x+a}\right)=m+n.$

(2.) $\left.\begin{array}{r} x-y=12 \\ x^2+y^2=74 \end{array}\right\}.$

7. If $r\,(la+mb)=ab=r'\,(ma+lb)$, and $l+m=1$;

$$\text{then } \frac{1}{r}+\frac{1}{r'}=\frac{1}{a}+\frac{1}{b}.$$

8. The area of an ellipse varies as the product of its axes. When the axes are equal the ellipse becomes a circle, and if the radius is 1, the area is equal 3.14159: what is the area of an ellipse whose axes are 3 and 5?

9. The first, second, and third terms of a geometric progression are the same as the first, third, and fourth terms of an arithmetic series. Find the common ratio and the common difference.

10. If $m+x$, $2m$, $m+y$, be in harmonical progression, then x, m, y are in geometric progression.

VII.

1. Multiply together

$$1+x,\ 1-x+x^2-x^3,\ 1+x^4,\ 1+x^8.$$

2. Divide $a^{m+n}-a^mb^n+a^nb^m-b^{m+n}$ by a^n-b^n, and multiply the result by a^m-b^m.

3. What is the greatest common measure of

$$ax^2-a^2x,\ a^2x^2+ab^2x, \text{ and } abx^2+a^3x?$$

4. Find the least common multiple of

$$x^3-x,\ x^3-1, \text{ and } x^3+1.$$

5. Simplify the expression

$$\frac{3}{2(a^2-b^2)}\left\{\frac{1}{4}\frac{b^2}{a^2}(2ax-x^2)^2-\frac{b^2}{a}(ax^2-\frac{1}{3}x^3)+\frac{2}{3}ax^3-\frac{1}{4}x^4\right\}.$$

6. Solve the equations:

$$(1.)\quad (a-x)\,\frac{x+m}{x+n}=(a+x)\,\frac{x-m}{x-n}.$$

$$(2.)\quad \alpha x+\frac{m}{\alpha}=y=\beta x+\frac{m}{\beta}.$$

7. If $x+y+z=h$, and $kx=ly=mz$, then each of these latter quantities $=\dfrac{h}{\frac{1}{k}+\frac{1}{l}+\frac{1}{m}}$.

8. Show that two snowballs of 8 and 16 inches diameter will, when melted, cover the bottom of a tub 24 inches in diameter to the depth of $5\frac{1}{3}$ inches.

N.B. Four times the number of square inches in a circle of one inch radius = three times the number of cubic inches in a sphere of the same radius.

9. If $\dfrac{a-a'}{b-b'}+\dfrac{aa'}{bb'}=0$; then a, b and a', b' have the same harmonic mean.

10. Sum the series

$1+(r+\rho)+(r^2+r\rho+\rho^2)+(r^3+r^2\rho+r\rho^2+\rho^3)+$ &c. *ad infinitum.*

VIII.

1. Multiply $x^4-(n-1)\,a^2x^2+a^4$ by x^2-a^2, and x^m+y^m by x^n-y^n.

2. Divide $x^4-px^3+qx^2-rx+s$ by $x-a$.

3. Find the least common multiple of

$$x - 1,\ 2x - 1,\ x^3 - 1,\ x^2 + x + 1,$$

and the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

4. Simplify

$$\frac{1}{4}\left\{\frac{1}{(x+1)^3} - \frac{1}{(x+1)^2}\right\} - \frac{1}{16}\left\{\frac{1}{x+1} - \frac{1}{x-1}\right\} + \frac{1}{8}\,\frac{1}{(x-1)^2}.$$

5. If $(2cx - a^2 - x^2 - y^2)^2 = \{x^2 + (a-y)^2\}\{x^2 + (a+y)^2\}$,

$$\text{then } \frac{y^2}{cx} + \frac{x^2}{cx - a^2} = 1.$$

6. Solve the equations:

(1.) $\dfrac{x}{x-1} = \dfrac{3}{2} + \dfrac{x-1}{x}$. (2.) $\left.\begin{aligned} x + y &= x^2 \\ 3y - x &= y^2 \end{aligned}\right\}$

(3.) $x + y - z = 8x + 3y - 6z = 3z - 4x - y = 1.$

7. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1lb. a head; after being at sea 20 days she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$lb.: find the original number of the crew.

8. The force of terrestrial gravity varies inversely as the square of the distance from the earth's centre. Compare its intensity at the earth's surface and at the moon, which is sixty of the earth's radii distant from the earth's centre.

9. In an arithmetic series, if the $(p + q)^{\text{th}}$ term $= m$, and the $(p - q)^{\text{th}}$ term $= n$: then the p^{th} term $= \frac{1}{2}(m + n)$, and the q^{th} term $= m - \frac{1}{2}(m - n)\dfrac{p}{q}$.

10. If σ_1 be the sum, σ_2 the sum of the squares, σ_3 the sum of the cubes of the terms of an infinite geometric series; then

$$\frac{\sigma_1 \sigma_3}{\sigma_2^2} + \frac{1}{3}\frac{\sigma_3}{\sigma_1^3} - \frac{4}{3} = 0.$$

IX.

1. Prove that

$$(x^2 + 3x + 1)^2 - 1 = x(x+1)(x+2)(x+3).$$

2. Show that

$$\left(x + \frac{1}{x}\right)^2 - \left(y + \frac{1}{y}\right)^2 = \left(xy - \frac{1}{xy}\right)\left(\frac{x}{y} - \frac{y}{x}\right).$$

3. Show that

$$\frac{1}{1 + x^{m-n} + x^{m-p}} + \frac{1}{1 + x^{n-m} + x^{n-p}} + \frac{1}{1 + x^{p-m} + x^{p-n}} = 1.$$

4. Find the value of

$$\frac{1}{2(x+1)} - \frac{9}{25} \cdot \frac{1}{x+2} - \frac{1}{5} \cdot \frac{1}{(x+2)^2} - \frac{1}{50}\frac{7x+1}{x^2+1}.$$

5. Solve the equations:

$$(1.)\ \frac{1}{x-2} - \frac{2}{x+2} = \frac{3}{5}. \quad (2.)\ \frac{x^2 + xy}{15} = \frac{xy - y^2}{2} = 1;$$

and show that the value of $\frac{s}{f}$ obtained from the equation

$$\frac{3}{(\mu - 1)^2}\left\{\frac{1}{f} + \frac{\mu - 1}{s}\right\}^2 - \left(\frac{1}{s} - \frac{1}{f}\right)\left(\frac{3}{s} - \frac{2\mu + 3}{f}\right) = 0$$

$$\text{is } \frac{2(\mu - 1)(\mu + 2)}{2\mu^2 - \mu + 4}.$$

6. The hour, minute, and second hands of a watch all turn on the same centre. When will the second hand first bisect the angle between the other two, after five minutes past 12?

7. If $y = ax + \frac{m}{a}$, and $y = \frac{1+a}{1-a} \cdot x$;

$$\text{then } (y - x)(x^2 + y^2) = m(x + y)^2.$$

8. Prove the following proposition algebraically:

If four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together. (*Euclid*, Book V. 25.)

9. If S denote the sum of n^2 terms of natural numbers, and s_r denote the sum of n terms of the series formed by taking the first and every r^{th} term of the odd numbers 1, 3, 5, &c.; then

$$S = s_1 + s_2 + s_3 + \&c. \dots + s_n.$$

10. If a_1, a_2, a_3, a_4, be in harmonical progression; then

$$\frac{a_4}{a_3} = \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{a_2}{a_1}}}$$

X.

1. Find the difference between $a(b+c)^2 + b(a+c)^2 + c(a+b)^2$, and $(a+b)(a-c)(b-c) + (a-b)(a-c)(b+c) - (a-b)(b-c)(a+c)$.

2. Simplify the expression:

$$3a - [b + \{2a - (b - x)\}] + \frac{1}{2} - \frac{\frac{1}{2} - 2x^2}{2x+1};$$

and find the cube root of $1 - 3x + 5x^3 - 3x^5 - x^6$.

3. Prove that

$$1 - \left\{\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}\right\}^2 = \frac{4(s-a)(s-b)(s-c)(s-d)}{(ab+cd)^2}$$

where $\qquad 2s = a + b + c + d.$

4. Add together

$$\frac{9}{x-2},\ \frac{11}{2(3-x)},\ \frac{5}{2(1-x)};$$

and show that if $m = \dfrac{a^2+b^2}{a-b}$, $n = \dfrac{a^2+b^2}{2\sqrt{ab}}$;

then $(a+b)^2 - \dfrac{m^2n^2}{m^2+n^2} = 4ab \cdot \dfrac{a^2+ab+b^2}{(a+b)^2}$.

5. Solve the equations:

(1.) $a - x = \sqrt{2ax+x^2}$.

(2.) $$\left.\begin{aligned} \frac{bx+ay}{a} &= \frac{xy}{c} \\ \frac{ax-by}{b} &= \frac{xy}{c} \end{aligned}\right\}.$$

6. If $\dfrac{x}{a} + \dfrac{y}{b'} = 1$, $\dfrac{x}{a'} + \dfrac{y}{b} = 1$, and $a' + b' = a + b$, then $x + y = a + b$.

7. Eliminate m between the equations:

$$\left.\begin{aligned} y &= mx + \sqrt{m^2a^2+b^2} \\ my + x &= \sqrt{a^2-b^2} \end{aligned}\right\}.$$

8. A can do a piece of work in p days, B the same in q days, and C in $\frac{1}{2}(p+q)$ days. How long will they take to do it working all together?

9. Find the sum of the n^{th} terms of n arithmetic series whose first terms are respectively 1, 2, 3, . . . and common differences 1, 3, 5, ...

If the first terms and common differences are interchanged, and the new sum be s'; find the value of n, when

$$s : s' :: 11 : 7.$$

10. If the limits of a series of geometric progressions are in harmonic progression, their common ratios are in arithmetic progression.

XI.

1. Divide $x^{2m} - (xy)^{m+n} + (xy)^{m-n} - y^{2m}$ by $x^{m+n} + y^{m-n}$.

2. Show that whatever whole number m be, $x^m - y^m$ is always exactly divisible by $x - y$.
When is $x^m + y^m$ exactly divisible by $x + y$?

3. Resolve into factors $x^2 + x - 6$, $x^2 - x - 6$, and find the least common multiple of $x^2 - 2x - 3$, $x^2 + 2x - 15$.
Reduce to lowest terms

$$\frac{(a^2 - 4b^2)(a^2 + ab - 2b^2)}{(a^2 - b^2)(a^2 - ab - 2b^2)}.$$

4. Simplify the expression

$$\left\{\frac{2x-5}{(x+3)(x+1)^2} + \frac{7}{2(x+1)^2}\right\} \div \left\{\frac{1}{x+1} - \frac{1}{x+3}\right\};$$

and show that

$$\frac{(1-a^2)(1-b^2)(1-c^2) - (c+ab)(b+ac)(a+bc)}{1 - a^2 - b^2 - c^2 - 2abc} = 1 + abc.$$

5. Solve the equations:

(1.) $a + x + \sqrt{a^2 + bx + x^2} = b.$

(2.) $\dfrac{7+x}{7-x} + \dfrac{7-x}{7+x} = \dfrac{29}{10}.$

(3.) $\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= a \\ \frac{1}{x} + \frac{1}{z} &= b \\ \frac{1}{y} + \frac{1}{z} &= c \end{aligned}\right\}$

6. If $y - mx = \sqrt{m^2 a^2 + b^2}$,
and $my + x = \sqrt{a^2 + m^2 b^2}$,
then $x^2 + y^2 = a^2 + b^2$.

7. If $a : b :: c : d$, and $a' : b' :: c' : d'$,

then $aa' : bb' :: cc' : dd'$,

and $\frac{a}{a'} : \frac{b}{b'} :: \frac{c}{c'} : \frac{d}{d'}$.

8. The planets perform their revolutions in times the squares of which vary as the cubes of their distances from the sun. Jupiter's periodic time is 11·85 years: show that he is very nearly $5\frac{1}{5}$ times as far from the Sun as the Earth.

9. If r, ρ be the common ratios of two infinite geometric series, such that the sum of each is equal to the first term of the other; then

$$\frac{1}{r} + \frac{1}{\rho} = 1.$$

10. There are two arithmetic series, the first term and common difference of one being the reciprocals of those of the other, and the product of their sums to n terms $= \frac{n^3}{4}(n+3)$. Find the ratios of the first terms to the common differences.

ERRATA.

Page 6, line 24, for $-\ d$, read $+\ d$.

Page 26, line 4, for $(a^2 - x)$ read $(a^2 - x^2)$.

Page 27, last line, for $1 - ba$ read $1 + ba$.

Page 52, line 27, read $ma + nb : mc + nd :: pa - qb : pc - qd$.

Page 53, line 23, omit the words "*sets of*."

Page 72, line 25, for $+\ c^2$, read $-\ c^2$.

Page 79, to Question 7, Ex. XV. add the words "*Explain your result.*"

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